Heterogeneous Beliefs, Speculation, and the Equity Premium

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ABSTRACT

Agents with heterogeneous beliefs about fundamental growth do not share risks perfectly but instead speculate with each other on the relative accuracy of their models' predictions. They face the risk that market prices move more in line with the trading models of competing agents than with their own. Less risk-averse agents speculate more aggressively and demand higher risk premiums. My calibrated model generates countercyclical consumption volatility, earnings forecast dispersion, and cross-sectional consumption dispersion. With a risk aversion coefficient less than one, agents' speculation causes half the observed equity premium and lowers the riskless rate by about 1%.

*Haskayne School of Business, University of Calgary. I thank an anonymous referee, an associate editor, Rob Stambaugh (the Editor), Suleyman Basak, Brian Boyer (EFA Discussant), Michael Brennan, Benjamin Croitoru (FEA Discussant), Kent Daniel, Darrell Duffie, Bob Goldstein, John Heaton, Joe Ostroy, Anna Pavlova, Chris Telmer (AFA Discussant), Pietro Veronesi, Luis Viceira, Harold Zhang (FRB Conference Discussant), and especially Phil Dybvig, as well as seminar participants at Washington University in St. Louis, London Business School, Lancaster University, University of Calgary, University of South Carolina, University of Central Florida, Federal Reserve Bank of St. Louis, Brigham Young University, the 2004 Annual Derivatives Securities Conference, Third World Congress of the Bachelier Finance Society, the 2004 European Finance Association, the 2004 Annual Conference on Financial Economics and Accounting, the 2005 Conference on Risk Premiums at the Federal Reserve Board, and the 2005 American Finance Association for helpful comments. I gratefully acknowledge a research grant from the SSHRC. Appendices C and D are available at: http://homepages.ucalgary.ca/∼adavid.

The representative agent paradigm with identical agents fails to take into account the speculative behavior of different agents in the economy. Under this paradigm, aggregate measures of fundamentals are sufficient to measure the risks faced by agents. Given the smoothness of these aggregate variables, this paradigm requires individuals to be implausibly highly risk-averse to match the observed risk premium on stocks (as first noted in the seminal paper by Mehra and Prescott (1985)). Moreover, it implies that agents assign very low values to safe assets, that is, they require a high riskless rate to be induced to clear bond markets.

An alternative viewpoint is that individuals process new information and decide on their trades using competing models of economic fundamentals. In contrast to the identical agent case, such individuals do not always attempt to share risks with each other; rather, in periods of large disagreement of views they take bets on the relative accuracy of their models' predictions. In addition to the fundamental risk in assets, they face the risk that market prices move more in line with the trading models of other agents than with their own. Speculative strategies by individuals therefore introduce new risk factors that affect premiums on all assets, namely, the beliefs of each trading type.[1] Because gains and losses from speculation are on net zero-sum, they are not captured in the statistics of aggregate data, which therefore underestimate the amount of risk faced by agents.

I construct a general equilibrium exchange economy in which two types of agents have heterogeneous beliefs. These agents agree to disagree on the parameters of fundamental processes in the economy and update their beliefs about the state of the economy differently even though they observe the same data. While I exogenously specify the models of different agents, their trading profits and survival are endogenous: Agents trading with models that do not fit the data well incur losses to agents trading with better models, and their share of the market declines over time. To simplify the discussion below, I will say that the two types of agents have different models of the fundamental process, even though in my analysis they simply disagree on a common model's parameter values. In equilibrium, they agree on the prices of assets but disagree on the decomposition of asset returns into their expected return and shock components. Due to this disagreement, agents speculate with each other and thus, as in models of incomplete markets, individuals' consumptions are not perfectly correlated.[2] At times of higher dispersion in beliefs *and* lower model disagreement the premium agents demand to hold stocks is larger.

While the effect of dispersion on the risk premium is intuitive, the role of model disagreement merits further clarification. The "disagreement value" as I refer to it, in equilibrium is the ratio of the two agents' state-price densities (price per unit probabilities). Since agents agree on all prices, this value is the ratio of the relative likelihood that the current data on fundamentals are generated by the model of type 2 agents (for brevity, I simply refer to these agents collectively as agent 2) rather than the model of type 1 agents (agent 1). The level of this variable is shown to depend on the past performance of the two agents' models. After a period in which the models perform comparably, the ratio is near its mean value (i.e., there is low model disagreement), agents' consumptions become nearly equal, and each agent has a large exposure to price movements due to the beliefs of the other type. Conversely, after a period of dominance by, say, agent 2, the ratio becomes large, agent 2 consumes a larger share of output, and the price variability due to agent 1's beliefs declines. Therefore, in periods of high model disagreement, there is low exposure to speculative risks.

I calibrate my model to fundamental data (aggregate earnings and consumption) and a series of dispersion of earnings forecasts obtained from the Survey of Professional Forecasters. I extend existing methodologies of the maximum likelihood procedure for unobservable regime-switching models Hamilton(1989,1994) to a Generalized Method of Moments (GMM) method that can estimate heterogeneous parameters for two groups of agents. As part of the procedure, I estimate a set of beliefs at each date that agents of each type hold of fundamentals being in a strong growth phase. The procedure puts weight on the likelihood of each type of agent observing the fundamental data, as well as on the dispersion in beliefs across the two groups.

(Insert Figure 1 about here)

In the model, there are two important properties of speculation and the risk it causes for individuals. First, the amount of risk faced by individuals is *endogenous.* Thus, in contrast to Mehra and Prescott (1985) and more recent complete market models that study the equity premium, in my model the risk premium on stocks does not monotonically increase in agents' risk aversion. As can be seen in the left panel of Figure 1, when agents' coefficient of relative risk aversion (CRRA) is less than one, the risk premium and CRRA are negatively related. The intuition for this finding is quite simple: Less risk averse agents speculate more aggressively. Further, as can be seen in the right panel, my model predicts that per capita consumption volatility increases as risk aversion declines. For low levels of risk aversion, the increase in the amount of risk dominates the decrease in the market price of risk, generating a higher risk premium on net.

Second, speculative activity undergoes sharp bursts of intensity in periods of weak fundamentals since the two agents disagree most strongly on the transition probability out of a recession state. In such periods, per capita consumption volatility and stock volatility both expand to deliver a double impact on the risk premium through the market price of risk and the amount of risk to be priced. This positive time-series covariation is important for my calibrated model to generate a large average risk premium with empirically plausible levels of average per capita consumption volatility. The bursts of speculative activity generate a countercyclical cross-sectional standard deviation of consumption growth across agents, consistent with recent empirical evidence by Storesletten, Telmer, and Yaron (2004). The calibrated model inherits this property from the dispersion series from surveys, which is strongly countercyclical.

Note that for each level of risk aversion plotted in Figure 1, the volatility of aggregate consumption is held constant at 1% per quarter, as is standard in the literature. This is the main binding constraint in the equity premium literature. However, recent evidence (Attansio, Banks, and Tanner (2002), Brav, Constantinides, and Geczy (2002)) suggests that the per capita volatility of quarterly consumption growth ranges between 6% and 12%. As can be seen, my model is capable of generating such levels of heterogeneity in consumption growth for levels of the CRRA smaller than 1. Finally, as the left panel of the figure shows, the average riskless rate in my model is fairly low for levels of the CRRA around 0.5, but climbs to implausible levels when the CRRA increases beyond 1. This is the risk-free rate puzzle as first documented in Weil (1989). For extreme levels of the CRRA in excess of 50, the riskless rate falls again (not shown in the plot).

In their seminal article, Mehra and Prescott (1985) suggest the lack of perfect insurability due to incomplete markets as a promising direction for the resolution of the equity premium puzzle. Mankiw(1986) follows this direction. However, subsequent articles suggest that if agents are allowed to trade in stocks and bonds, they are able to diversify away most of their idiosyncratic risk, since in equilibrium asset prices respond to changes in the aggregate income distribution (Telmer(1993), Lucas(1994), Heaton and Lucas(1996)). Constantinides and Duffie(1996) are able to resolve most aspects of the puzzle with permanent idiosyncratic shocks in an economy in which, in equilibrium, individuals are content to not trade and thus are unable to hedge the idiosyncratic shocks. My model has three significant differences from this literature: First, agents do not receive idiosyncratic shocks. Instead, there is idiosyncratic variation in their beliefs. As in the case of incomplete markets, agents' marginal rates of substitution in equilibrium are not equated because of their inability or unwillingness to insure each other from relative movements in beliefs. Second, individuals trade in equilibrium. Stock and bond prices depend on the distribution of beliefs. However, despite the consumption smoothing attained by trading in financial assets, differences in opinion remain, and hence agents continue to face an exposure to other agents' changing beliefs. Thus, despite trading, the model's equity premium does not shrink. Finally, trading losses that lead to endogenous shocks to consumption are not permanent. Such losses therefore do not cause a trend increase in the cross-sectional standard deviation of consumption growth across agents as in Constantinides and Duffie(1996).

Models of heterogeneous beliefs have become increasingly popular in recent years. Early writers in this field (seminal papers in the field include Lintner(1969), Williams(1979), Varian(1989), Harris and Raviv(1993), Kandel and Pearson(1995)) note that these models are able to generate patterns in trading volume because agents have differing opinions and agree to disagree after observing the same information.[3] In models based on asymmetric information, in contrast, agents' beliefs converge upon observing trades. None of these papers explicitly study the relationship between the equity premium and the time variation in agents' consumption moments, which is the subject of this paper. Dumas, Kurshev, and Uppal(2005) refer to the speculative risk of unexpected movements in competitors' beliefs modeled in this paper as "sentiment risk."

More directly, this paper extends the analysis of continuous time models of heterogeneous beliefs (Detemple and Murthy(1994), Zapatero(1989), Basak(2000), Basak and Croitoru(2000), Buraschi and Jiltsov(2002)) to the case of recurrent jumps in the underlying drift of diffusion process. Crucially, the dispersion process in these models declines monotonically over time and asymptotes to zero. Therefore, the dispersion of beliefs across agents has a temporary effect on the conditional risk premium, but is unable to match the large risk premium in long samples of data. In my analysis, however, agents have different underlying models of the data generating process as opposed to differing initial priors as in the above papers, and the dispersion process recurrently fluctuates and leads to a large equity premium over long horizons.

Before concluding this section, I compare my results with those of other authors using models in which agents have time-separable preferences and unrestricted access to capital markets. Reitz(1988) is able to generate a large equity premium with low risk aversion if agents price "peso problem" like events, such as a 70% drop in consumption. Longstaff and Piazzesi(2003) find some improvements — half the premium with a CRRA of five — with smaller jumps. Similarly, Bansal and Yaron(2004) price the risk from small but persistent changes in the expected growth rate of fundamentals and are able to justify half the observed premium with a CRRA of 7.5. In a paper related to this one, Brennan and Xia(2001) model the learning process of homogeneous agents about the dividend process and find that with a CRRA of 10, equity volatility assumes empirically plausible levels and they are able to generate about half the equity premium. Ait-Sahalia, Parker, and Yogo(2004) use a value of 7 to generate the entire risk premium in a model with multiple goods. With the exception of Reitz(1988), none of the other papers can lower the riskless rate. In comparison, in my calibrated model the Sharpe ratio attains a value in the range of about 9% to 15%, which is a little less than half the value for the aggregate U.S. stock market, stock volatility is generated at plausible levels of about 18%, and the equity premium is about 2.5% to 3% when agents have a CRRA in the 0.4 to 0.7 range. Moreover, the average riskless return is lower by almost 1% compared to a benchmark model with homogeneous beliefs and has a very low volatility as in the data.

The plan for the remainder of this paper is as follows. In Section I, I present the basic structure of the model. In Section II, I characterize the equilibrium and find approximate solutions for asset prices and portfolio choices. A calibration of the model is provided in Section III, and the performance of the model in addressing the equity premium puzzle is discussed in Section IV. I conclude in Section V. Appendix A covers essential proofs, while Appendix B extends the analysis of my model to the case in which stocks are in positive net supply. Two additional appendices, the first providing a detailed description of the projection method used to solve the PDE for asset prices, and the second providing the calibration methodology, are made available to readers upon request.

I. Structure of the Model

In this section, I introduce the assumptions of my economic setting.

ASSUMPTION 1: Dividends, q_t, evolve according to the log-normal process

$$\frac{dq_t}{q_t} = \theta_t \, dt + \sigma_q \, d\tilde{W}_t, \tag{1}$$

where $W_t = (W_{1t}, W_{2t})^\top$ is a two-dimensional vector of independent Weiner processes. The 1×2 constant vector σ_q is assumed to be known by all investors and is constant over time. The process for θ_t is described below.

ASSUMPTION 2: Total output in the economy, x_t, evolves according to the log-normal process

$$\frac{dx_t}{x_t} = \kappa_t \, dt + \sigma_x \, d\tilde{W}_t, \tag{2}$$

where the process followed by κ_t is described below and σ_x is a 1×2 constant vector that is known by investors.

It is convenient to stack together the "observation" processes (1) and (2). Let $y = (q, x)^\top$. We then have

$$\frac{dy_t}{y_t} = \nu_t \, dt + \Sigma \, d\tilde{W}_t,$$

where $\frac{dy_t}{y_t}$ is interpreted as "element-by-element" division, $\nu_t = (\theta_t, \kappa_t)^\top$, and $\Sigma = (\sigma_q^\top, \sigma_x^\top)^\top$. I assume that Σ is invertible.

ASSUMPTION 3: The drift vector ν_t follows a two-state, continuous time, finite state Markov chain with generator matrix Λ, that is, over the infinitesimal time interval of length dt, $\lambda_{ij} dt =$ prob $(\nu_{t+dt} = \nu_j | \nu_t = \nu_i)$, for $i \neq j$, and $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. The transition matrix over any finite interval of time, s, is $\exp(\Lambda s)$.

Assumptions 1 through 3 imply that real dividends and output follow a joint log-normal model with drifts that jump intermittently between two states. I provide some specification tests for the number of states in Appendix D. Following Cecchetti, Lam, and Mark(1990) and Brennan and Xia(2001), dividends are modeled as part of the entire output of the economy. Therefore, the claim to dividends, the stock, is in zero net supply.[4] Aggregate (across agents) consumption in the economy equals aggregate output. Agents can take "bets" on the stock price as a vehicle for risk sharing.

ASSUMPTION 4: There are $M = 2$ classes of investors. All agents have time-separable utility functions over infinitely dated stochastic consumption streams:

$$U(c) = E^{(m)}[\int_0^{\infty} \exp(-\rho s) \cdot u(c_s) ds],$$

with time discount factor ρ and felicity $u(c_t) = c_t^{\gamma}/\gamma$. The felicity function $u(.)$ has a constant coefficient of relative risk aversion $1-\gamma$, and satisfies the Inada conditions $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$.

ASSUMPTION 5: Agents of type m are collectively endowed with a constant fraction $0 < e^{(m)} < 1$ of output x_t in the economy at period t, where $e^{(1)} + e^{(2)} = 1$. The endowment at time t is $\epsilon_t^{(m)} = e^{(m)} x_t$, where x_t follows the process in Assumption 2.

ASSUMPTION 6: Individuals can trade in a short-term (instantaneous) riskless security and two long-term securities, stocks and consol bonds. Stocks pay a continuous dividend described by the process in Assumption 1. Bonds pay a continuous coupon of c per instant. The prices of stocks and bonds at time t are denoted by P_t and B_t respectively. Investors' portfolio choices in these assets are given by w_{P_t} and w_{B_t}, respectively. I impose constraints on admissible strategies that prohibit negative wealth at any future date (see Dybvig and Huang(1989)). Both stocks and bonds are in zero net supply. Note that the number of long-term assets equals the number of stochastic shocks driving the economy, so markets are dynamically complete.

The last assumption makes it possible to obtain fluctuating aggregate uncertainty and dispersion, which are the objects of this paper's investigation.

ASSUMPTION 7: Investors do not observe the realizations of the drifts, ν_t. An investor of class m estimates that the vector of drift parameters, ν, equals $\nu^{(m)}$, and the generator matrix, Λ, equals $\Lambda^{(m)}$. Investors of each type know the parameter values of all other types.

Investors learn about the drifts from observations of fundamentals. However, they agree to disagree about the evolution of states since they have different "models" of the state processes. Their perceptions are captured in different filtrations, $\mathcal{F}_t^{(m)}$, and probability measures, $\mathcal{P}_t^{(m)}$, on the drift states of fundamentals. In the standard heterogeneous beliefs framework (for example, Basak(2000)) agents disagree on the prior distribution of relevant state variables. My assumption is similar to that in Harris and Raviv(1993), in which investors have different likelihood functions of the relationship between observed signals and returns on assets, and each investor is absolutely convinced that his model is correct. It is also a form of the overconfidence model of Daniel, Hirshleifer, and Subrahmanyam(2001), in which each investor places an excessively large weight on his personal model. Given the observation of y_t, investors form the posterior probability $\pi_{1t}^{(m)} = \text{prob}(\nu_t = \nu_1^{(m)} | \mathcal{F}_t^{(m)})$ of fundamentals being in state 1 at time t. I denote conditional means with bars, for example, $\overline{\nu}_t^{(m)} = \sum_{i=1}^{2} \pi_{it}\, \nu_i^{(m)}$.

LEMMA 1: *Given an initial condition* $0 \leq \pi_{10}^{(m)} \leq 1$, *the probabilities* $\pi_{1t}^{(m)}$ *follow the stochastic differential equations*

$$d\pi_{it}^{(m)} = \mu_{it}^{(m)} dt + \sigma_i^{(m)} d\tilde{W}_t^{(m)}, \tag{3}$$

where

$$\mu_{1t}^{(m)} = (\lambda_{12}^{(m)} + \lambda_{21}^{(m)})[\pi_1^{*(m)} - \pi_{1t}^{(m)}], \tag{4}$$

$$\sigma_{1t}^{(m)} = \pi_{1t}^{(m)}(1 - \pi_{1t}^{(m)})(\theta_1^{(m)} - \theta_2^{(m)}, \kappa_1^{(m)} - \kappa_2^{(m)}) \cdot (\Sigma^{\top})^{-1}, \tag{5}$$

$$d\tilde{W}_t^{(m)} = \Sigma^{-1}\left(\frac{dy_t}{y_t} - E_t^{(m)}[\frac{dy_t}{y_t}]\right) = \Sigma^{-1}(\nu_t - \overline{\nu}^{(m)})dt \; + \; d\tilde{W}_t. \tag{6}$$

Proof: See Wonham(1964) or David(1993).

The first application of this result in financial economics as well as several properties of the filtering process are derived in David(1997). In particular, $\pi_{1t}^{(m)}$ mean reverts to its unconditional mean, $\pi_1^{*(m)} = \lambda_{21}^{(m)}/(\lambda_{12}^{(m)} + \lambda_{21}^{(m)})$, with a speed proportional to $(\lambda_{12}^{(m)} + \lambda_{21}^{(m)})$, and the volatility of an agent of type m's updating process is the product of his uncertainty, $\pi_{1t}^{(m)}(1 - \pi_{1t}^{(m)})$, and the signal-to-noise ratio, $(\theta_1^{(m)} - \theta_2^{(m)}, \kappa_1^{(m)} - \kappa_2^{(m)}) \cdot (\Sigma^\top)^{-1}$.

For later reference, I rewrite the fundamental process as $\frac{dy_t}{y_t} = \bar{\nu}_t^{(m)} dt + \Sigma d\tilde{W}_t^{(m)}$, where $d\tilde{W}_t^{(m)}$ is an "innovations" process under the filtration of agent m. Under the separation principle it can be used for dynamic optimization (see David(1997) for a discussion). The difference between the two agents' innovation processes is given by

$$d\tilde{W}_t^{(2)} \; = \; d\tilde{W}_t^{(1)} + \sigma_{\eta t}\, dt, \tag{7}$$

where $\sigma_{\eta t} = \Sigma^{-1}(\bar{\nu}_t^{(2)} - \bar{\nu}_t^{(1)})$. As can be seen, agents' estimated switching probabilities between drift states, $\lambda_{ij}^{(m)}$, can be substantially different, and yet $d\tilde{W}_t^{(2)} - d\tilde{W}_t^{(1)}$ is of the order $O(dt)$.

Let $\nu^{*(m)} = \pi_1^{*(m)}\, \nu_1^{(m)} + (1 - \pi_1^{*(m)})\, \nu_2^{(m)}$ be the unconditional mean of fundamental growth assessed by investor m. I do not require that $\nu^{*(1)} = \nu^{*(2)}$; that is, the unconditional mean estimates of the two agents may differ. Since the parameter differences affect only drift rates, I show in the following corollary that the probability measures of the two agents are equivalent over any finite interval.

COROLLARY 1: *The restriction of agents' probability measures $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ to the filtration $\mathcal{F}_t$ at time t, $\mathcal{P}_t^{(1)}$ and $\mathcal{P}_t^{(2)}$, are equivalent for all $t \in [0, \infty)$. The Radon-Nikodym derivative of $\mathcal{P}_t^{(2)}$ with respect to $\mathcal{P}_t^{(1)}$ is given by $\varrho_t = \varrho_0 \exp\left(-1/2 \int_0^t \sigma_{\eta s}^\top \sigma_{\eta s} ds \, + \, \int_0^t \sigma_{\eta s} d\tilde{W}_t^{(1)}\right)$, which is a martingale with respect to $\mathcal{P}_t^{(1)}$ on the time interval $[0, t]$ for all t.*

Proof: See Appendix A

Corollary 1 implies that the two probability measures are equivalent on the filtrations of agents at any finite time t. The measures may be mutually singular over the the infinite horizon when the long-term mean drifts of the two agents are not equal.[5] I show in the next section that the mutual singularity does not preclude agreement on security values by the two agents, nor does it imply the existence of arbitrage opportunities.

II. Market Equilibrium

A rational expectations equilibrium is a set of utility-maximizing consumption choices for each agent and a set of conjectured prices for all securities in each date and state for each agent so that total consumption equals total output in the economy, markets clear, and agents agree on prices in all dates and states. Due to the existence of two long-lived securities, markets are dynamically complete and ensure the existence of unique Arrow-Debreu (A-D) security prices for each agent under his own filtration. In equilibrium, agents agree on these A-D prices as well.

I first examine individuals' consumption choice problems. Each agent maximizes the utility function in Assumption 4 subject to the budget constraint

$$E^{(m)}\left[\int_0^\infty c_s^{(m)}\xi_s^{(m)}ds\,\right] \leq E^{(m)}[\int_0^\infty \epsilon_s^{(m)}\xi_s^{(m)}ds\right] \equiv X_0^{(m)}, \tag{8}$$

where $\xi_t^{(m)}$, his state-price density (SPD) function for consumption at t, is determined endogenously in equilibrium and $X_0^{(m)}$ is the value of his endowment as specified in Assumption 5 at period 0. The necessary conditions for optimality (see Karatzas, Lehoczky, and Shreve(1987), Cox and Huang(1989)) are $u'(c_t^{(m)}) = y_m\xi_t^{(m)}$ for each m. [6]

Using the SPD, I can write the pricing kernel for an investor of type m as

$$\frac{d\xi_t^{(m)}}{\xi_t^{(m)}} \;=\; -r_t dt - \phi_t^{(m)} d\tilde{W}_t^{(m)}, \tag{9}$$

in which the real rate of interest, r_t, and market prices of risk, $\phi^{(m)}$, are determined endogenously. Given the pricing kernel in equation (9), the equilibrium price[7] of a traded security i with a non-negative payout flow $\{\delta_{it}\}$ is determined by individuals of type m as

$$\xi_t^{(m)} P_{it} \; = \; E_t^{(m)} \left[\int_t^{\infty} \xi_s^{(m)} \, \delta_{is} \, ds | \mathcal{F}_t^{(m)} \right]. \tag{10}$$

For equilibrium to exist, agents must agree on the level of prices at each date and state. This requirement puts restrictions on securities' risk premiums under the measures of the different agents and the objective measure, which I provide below. For security i, I can write the dynamics of the price process under the objective measure as

$$\frac{dP_{it}}{P_{it}} \; = \; \mu_{it} dt \; + \; \sigma_{it} d\tilde{W}_t, \tag{11}$$

or in terms of the information filtration of agent m, as

$$\frac{dP_{it}}{P_{it}} \; = \; \bar{\mu}_{it}^{(m)} dt \; + \; \sigma_{it} d\tilde{W}_t^{(m)}. \tag{12}$$

Using the definition of $\tilde{W}_t^{(m)}$ in (6), agreement by all agents on the level of prices at each date implies that

$$\mu_{it} - \bar{\mu}_{it}^{(m)} \; = \; \sigma_{it} \Sigma^{-1} (\nu_t - \bar{\nu}_t^{(m)}) \tag{13}$$

for each m, a relationship that I will examine more closely in Section IV. In addition, the expected returns of the two different agents are related by

$$\bar{\mu}_{it}^{(1)} - \bar{\mu}_{it}^{(2)} = \sigma_{it} \Sigma^{-1} (\bar{\nu}_t^{(1)} - \bar{\nu}_t^{(2)}). \tag{14}$$

I now provide a condition under which agents agree on prices.

PROPOSITION 1: *Agents agree on the level of prices at all dates and states if and only if*

$$(\phi_t^{(1)} - \phi_t^{(2)})^{\top} \; = \; \Sigma^{-1} (\bar{\nu}_t^{(1)} - \bar{\nu}_t^{(2)}). \tag{15}$$

The proof is in Appendix A.

It is relevant at this stage to point out why agents agree on prices despite having different probability measures over states of fundamental growth. Essentially, investors take bets on states of fundamental growth, "trading away" consumption from states that they think are less likely. Agents' first-order condition for optimization, $u'(c_t^{(m)}) = \xi_t^{(m)}$, implies that state-contingent dividends are priced so that large marginal utility compensates for a state's small probability. Differences in unconditional drift rates are compatible with these valuations: At long horizons, each agent consumes nearly the whole endowment in states in which the realized mean is close to the agent's believed long-term mean but far from what other agents believe, since for agents with constant relative risk aversion, no agent can have negative consumption.

A key state variable in my analysis is the disagreement value process, $\eta_t = \xi_t^{(1)}/\xi_t^{(2)}$, which is the ratio of the two agents' SPDs. Since an SPD is the state price per unit probability assessed by an agent, and agents agree on all prices (including A-D state prices), η_t is the ratio of the likelihood of observing the fundamentals at date t as a realization of the model of agent 2 to that of agent 1. Therefore, η_t increases when the observed fundamental data at date t are more likely to arise from the model of agent 2 rather than from that of agent 1. In periods when η_t is close to its mean value, there is "low disagreement" since the fundamentals are equally likely to be a realization of either agent's model. Conversely, η_t is far from its mean value when fundamental news supports one agent's model over that of the other. In the extreme cases, when news completely supports the model of type 1(2), $\eta_t \to 0(\infty)$. Proposition 1 implies that

$$\frac{d\eta_t}{\eta_t} = (\bar{\nu}_t^{(2)} - \bar{\nu}_t^{(1)})^\top \Sigma^{-1^\top} d\tilde{W}_t^{(1)} = \sigma_{\eta t}^\top d\tilde{W}_t^{(1)}. \tag{16}$$

This formulation enables me to study the evolution of the disagreement process given the history of each agent's beliefs.

As can be seen, η_t increases under two conditions: (i) When agent 1 has a positive surprise ($d\tilde{W}_t^{(1)} > 0$) and agent 2 is more optimistic than agent 1, or (ii) when agent 1 has a negative surprise and agent 2 is more pessimistic than agent 1. In the calibrated equilibrium discussed in the next section, I find that (i) holds in 57% of the sample. In such instances, positive surprises to fundamentals lend more support to the model of agent 2. In the remaining 43% of the sample, positive surprises to fundamentals lead to decreases in η and lend more support to the model of agent 1. The intuition of the sign of these effects is quite straightforward. For example, if agent 1 receives a positive surprise and agent 2 is more pessimistic than agent 1, then agent 1 would receive an even larger positive surprise, and as a result the likelihood that the data are a realization of agent 1's model would increase (η would decrease). This characterization explains why the correlation between agents' consumption growths and stock returns is time-varying, and in fact switches sign over time.

Proposition 2 below characterizes equilibrium consumptions, the riskless rate, and the market prices of risk of the two agents. Before presenting this proposition, I introduce the following lemma, which characterizes the consumption processes of the two agents.

LEMMA 2: *To be consistent with utility maximization, the consumption process of an individual of type m follows the diffusion process $dc_t^{(m)} = \mu_{ct}^{(m)} dt + \sigma_{ct}^{(m)} d\tilde{W}_t^{(m)}$, with volatility and drift coefficients*

$$\sigma_{ct}^{(m)} = \frac{1}{a_t^{(m)}} \phi_t^{(m)} \tag{17}$$

$$\mu_{ct}^{(m)} = \frac{1}{a_t^{(m)}} r_t + \frac{1}{2} \frac{b_t^{(m)}}{a_t^{(m)2}} \phi_t^{(m)} \phi_t^{(m)\top}, \tag{18}$$

where $a_t^{(m)} = -u_m''(c_t^{(m)})/u_m'(c_t^{(m)}) = (1-\gamma)/c_t^{(m)}$ *and* $b_t^{(m)} = -u_m'''(c_t^{(m)})/u_m''(c_t^{(m)}) = (2-\gamma)/c_t^{(m)}$.

Proof: See Appendix A.

The lemma shows that the volatilities of individuals' consumption growths are time-varying and equal the product of the inverse of the CRRA and the market prices of risk. Since the norm of the market prices of risk of agent m at any given time t equals the conditional maximal Sharpe ratio (as perceived by agent m) of all assets in the economy, the volatilities of individuals' consumption growths summarize the information about the conditional Sharpe ratios that are attainable by my model. As I show below, however, neither the volatility of aggregate consumption growth nor per capita consumption volatility are sufficient statistics for conditional Sharpe ratios. I look at $\phi_t^{(m)}$ carefully below.

PROPOSITION 2: *In equilibrium,*

(i) The individual consumption flow rates are

$$c_t^{(1)} = \frac{x_t}{1 + k\,\eta_t^{\frac{1}{1-\gamma}}} \tag{19}$$

$$c_t^{(2)} = \frac{k\,\eta_t^{\frac{1}{1-\gamma}}\,x_t}{1 + k\,\eta_t^{\frac{1}{1-\gamma}}}, \tag{20}$$

where $k = (y_1/y_2)^{\frac{1}{1-\gamma}}$.

(ii) The riskless rate in the economy is given by

$$\begin{aligned} r_t &= \rho - \frac{1}{2}\,(2-\gamma)\,(1-\gamma)\,\sigma_x\sigma_x^\top + \frac{1-\gamma}{1 + k\,\eta_t^{\frac{1}{1-\gamma}}}\left(\bar{\kappa}_t^{(1)} + k\,\eta_t^{\frac{1}{1-\gamma}}\,\bar{\kappa}_t^{(2)}\right) \\ &- \frac{\gamma\,k\,\eta_t^{\frac{1}{1-\gamma}}\,[(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})\sigma_x - (\bar{\kappa}_t^{(1)} - \bar{\kappa}_t^{(2)})\sigma_q]^2}{2\,(1-\gamma)\left(1 + k\,\eta_t^{\frac{1}{1-\gamma}}\right)^2 |\Sigma|^2}. \end{aligned} \tag{21}$$

(iii) Finally, the market prices of risk of the two types of agents are

$$\phi_q^{(1)} = (1-\gamma)\sigma_{x,1} + \frac{k\,\eta_t^{\frac{1}{1-\gamma}}}{1 + k\,\eta_t^{\frac{1}{1-\gamma}}}\,\frac{(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})\sigma_{x,2} + (\bar{\kappa}_t^{(2)} - \bar{\kappa}_t^{(1)})\sigma_{q,2}}{|\Sigma|}, \tag{22}$$

$$\phi_q^{(2)} = (1-\gamma)\sigma_{x,1} + \frac{1}{1 + k\,\eta_t^{\frac{1}{1-\gamma}}}\,\frac{(\bar{\theta}_t^{(2)} - \bar{\theta}_t^{(1)})\sigma_{x,2} + (\bar{\kappa}_t^{(1)} - \bar{\kappa}_t^{(2)})\sigma_{q,2}}{|\Sigma|}, \tag{23}$$

$$\phi_x^{(1)} = (1-\gamma)\sigma_{x,2} + \frac{k\,\eta_t^{\frac{1}{1-\gamma}}}{1+k\,\eta_t^{\frac{1}{1-\gamma}}} \frac{(\bar{\theta}_t^{(2)} - \bar{\theta}_t^{(1)})\sigma_{x,1} + (\bar{\kappa}_t^{(1)} - \bar{\kappa}_t^{(2)})\sigma_{q,1}}{|\Sigma|}, \tag{24}$$

$$\phi_x^{(2)} = (1-\gamma)\sigma_{x,2} + \frac{1}{1+k\,\eta_t^{\frac{1}{1-\gamma}}} \frac{(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})\sigma_{x,1} + (\bar{\kappa}_t^{(2)} - \bar{\kappa}_t^{(1)})\sigma_{q,1}}{|\Sigma|}. \tag{25}$$

Proof. See Appendix A.

It is important to note that the riskless rate and the market prices of risk characterized above depend on the constant $k = (y_1/y_2)^{\frac{1}{1-\gamma}}$. Therefore, the agents' SPD functions (equation 9) are also dependent on k. I determine k by ensuring that the budget constraints of the agents (equation 8) are satisfied with equality with the assumed SPDs. Since this step involves the solution of a partial differential equation (PDE), I defer its discussion to Section II.A. I provide a discussion of the qualitative features of the equilibrium below.

I first make some comments on the riskless rate in equation (21). The first two terms in this equation are standard. The first term reflects agents' time preference. The second term reflects the precautionary demand arising from the noise in the consumption process — with higher consumption volatility, agents' demand for riskless assets increases as they desire safer portfolios to offset risk, lowering the equilibrium real rate. The precautionary demand increases in the prudence of agents, captured by the term $(1-\gamma)\cdot(2-\gamma)$. The third term is the usual wealth effect on consumption: When the expected growth rate of consumption increases, agents are less willing to save for the future, leading to a higher equilibrium real rate. Here, the expected growth rates of the two agents are weighted by their respective shares of total consumption.

(Insert Figure 2 about here)

The last term in the interest rate expression represents the "hedging" demand term. Since this term is the product of two parts, each part must be large for an impact on the rate to obtain. The first part, $\left(k\,\eta_t^{\frac{1}{1-\gamma}}\right) / \left(1+k\,\eta_t^{\frac{1}{1-\gamma}}\right)^2$, is a concave function of η with a maximum at $\bar{\eta} = 1/k^{(1-\gamma)} = y_2/y_1$. From Proposition 2 (i), this part is the product of the agents' shares of consumption. When the relative performance of the two types of agents has been similar, $\eta \simeq \bar{\eta}$, that is, there is low model disagreement. At such times, each type of agent's share is near one-half, and each group of agents can potentially impact market prices. Therefore, agents face the risk that prices move with the beliefs of each type, or conversely, they perceive higher speculative opportunities during these times. Similarly, the second part represents speculative opportunities that arise from the dispersion in agents' beliefs. In periods of higher dispersion, the difference in expected growth rates increases. This effect is shown in the left panel of Figure 2 where I hold η constant and allow agents' beliefs of the expansion states to diverge. As I depart farther from the diagonal line, the dispersion in expected growth rates increases so that the riskless rate falls. In my model calibration, I find that the two agents' beliefs are highly correlated, but in periods of increased dispersion of beliefs, the rate drops significantly and helps to reduce the average riskless rate. In either case, low disagreement or high dispersion, agents' savings response depends on their CRRA. In contrast, for investors with CRRA larger than one, an improvement in opportunities makes them want to consume more currently due to a dominating wealth effect, causing a higher market clearing interest rate. Investors with CRRA less than one want to save and invest more currently due to a dominating substitution effect, leading to a lowering of the riskless rate.[8] Thus, to obtain a low riskless rate, I require that investors be of the latter type, that is, have a CRRA less than one.[9] Notably in this case, the impact of speculation is to decrease the riskless rate below that of a benchmark economy in which agents have homogeneous beliefs.

As can also be seen from equation (21), the short rate in an economy with heterogeneous beliefs depends not only on agents' current beliefs through the terms $\bar{\kappa}_t^{(m)}$ and $\bar{\theta}_t^{(m)}$, but also on their *lagged* beliefs through the disagreement value η_t. From (16), it is evident that η_t can be written as

$$\eta_t = \exp\left[\int_0^t -\frac{1}{2}\,(\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})^\top(\Sigma\Sigma^\top)^{-1}(\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})ds + \int_0^t \Sigma^{-1}(\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})d\tilde{W}_s^{(1)}\right], \quad (26)$$

that is, η_t is an integral of the weighted averages of past dispersions in the expected drifts of earnings and consumption. In other words, the short rate process displays path dependence with respect to past differences in opinions of the different agents. When agents have homogeneous beliefs, $\eta_t = 1$ at each time, the last term in (21) vanishes, and so does the path dependence. The path dependence of the short rate will be useful for obtaining a low volatility of interest rates in my model, a feature of the data.

Turning to the market prices of risk in part (iii) of Proposition 2, these consist of two terms, the risk in aggregate fundamentals, $(1-\gamma)\sigma_x$, and a term related to speculative risk. For a given agent, the second term itself is the product of two components, the share of the *other* agent's consumption of total output and a speculative component, which is proportional to the amount that an agent's expected growth of earnings exceeds that of the other agent ($(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})$ for agent 1). Both terms have intuitive meanings. The first component implies that when an agent consumes a smaller share of output, he faces greater exposure to prices moving in the direction of the other agent's beliefs. The second component implies that when an agent is more optimistic relative to the other, he faces a potentially larger correction of prices moving in line with the other agent's beliefs, and hence his price of risk increases.[10] Indeed, the second term is negative when an agent is more pessimistic than the other agent. Finally, as γ increases (agents are less risk averse), their consumption shares become more volatile, thereby giving larger weights to the dispersion terms in the speculative risk component. In Section IV I show that for a region of low risk aversion, this effect can dominate the first effect (risk in aggregate fundamentals), causing Sharpe ratios to increase for lower CRRA.

It is interesting to note that agents' market prices of dividend risk are nonzero despite stocks (that is, claims to future dividends) being in zero net supply. The intuition behind this result is that fluctuations in dividend growth lead to a divergence in opinions about *future* growth rates and speculative possibilities, which in turn lead to fluctuations in individual consumptions. Therefore, each agent demands a risk premium to bear these shocks. With zero dispersion in beliefs, this channel disappears.

A key feature of my model is its ability to generate high and persistent stock market volatility that arises endogenously as agents learn about fundamentals. When fundamental growth changes rapidly, agents' confidence in their current estimates declines, and their beliefs fluctuate rapidly with news. As David(1997) shows, such a learning-based volatility process satisfies many of the stylized facts in the GARCH literature. In this model, the speculation between agents has a further impact on volatility as displayed in Figure 2 (middle panel). Specifically, volatility increases in the uncertainty of each type, and in periods when both agents have maximum uncertainty (expansion probability of 0.5), volatility is 50% higher than in the case in which a representative agent has maximum uncertainty.

The equity risk premium for each agent is an inner product of the market prices of risk of that agent in Proposition 2 and equity volatility in equation (C6). The risk premium inherits many of the features of the price of risk discussed above, and thus I not repeat these discussions. Notably different, however, is the decline in risk premiums due to lower volatility periods of high certainty. In the right panel of Figure 2 I look at the effects of dispersion on the premium of agent 2. In periods when agent 2 is more optimistic about the state of fundamental growth, his risk premia are positive. In such periods, the premium of agent 1 is negative. As we see below, agent 2 takes long positions in stocks in most such periods, while agent 1 takes positions of the opposite sign.

A natural question that arises concerns the relationship between the conditional risk premium under the objective measure in the model and agents' beliefs. I summarize this relationship in the following corollary.

COROLLARY 2:

$$\mu_i - r = (1-\gamma)\sigma_x\sigma_i^\top + \left[\sum_{m=1}^{2} \frac{c_t^{(m)}}{x_t}\,\Sigma^{-1}\left(\nu_t - \nu_t^{(m)}\right)\right]\sigma_i^\top . \tag{27}$$

The proof is in Appendix A.

The premium under the objective measure is thus the sum of the premium in the benchmark economy with homogeneous beliefs and the consumption share-weighted estimation errors of the two agents. It follows at once that the conditional premium is $(1-\gamma)\,\sigma_x\,\sigma_i^\top$ if both agents are conditionally unbiased. The following case is particularly relevant in analyzing the premium in my calibrated economy in Section IV: $\Sigma^{-1}(\nu_t - \nu_t^{(1)}) = (\delta, 0)$ and $\Sigma^{-1}(\nu_t - \nu_t^{(2)}) = (0, 0)$. In this case, agent 1 is pessimistic about earnings growth and agent 2 is unbiased, in which case the premium is $(1-\gamma)\sigma_x\sigma_{it}^\top + \frac{c_t^{(1)}}{x_t}\,\delta\,\sigma_{i,1}$. The average premium under the objective measure in a long sample remains large only if the consumption share of agent 1 remains large despite trading using a biased estimate.[11]

A. Asset Prices and Portfolio Choices

Since there are two shock processes in the economy, agents require at least two multiperiod securities in addition to instantaneous bonds to complete the market (in the traditional sense of market completeness). I implement the equilibrium with a stock paying the dividend process in Assumption 1, and a consol (perpetuity) bonds paying a constant coupon flow c. I briefly describe the pricing of these securities below.

The prices are functions of each agent's beliefs and the disagreement value process. I use standard no-arbitrage analysis (see, for example, Cochrane(2001)) to value these securities. Since agents agree on the prices of all assets, I formulate the PDE under the filtration of agents of type 1. The stock price $P(\pi^{(1)}, \pi^{(2)}, \eta, q)$ is obtained by solving

$$E^{(m)}\left[\frac{dP}{P}\right] + \left(\frac{q}{P} - r(\pi_1^{(1)}, \pi_2^{(2)}, \eta)\right)\,dt \;=\; -E^{(m)}\left[\frac{dP}{P}\,\frac{d\xi^{(m)}}{\xi^{(m)}}\right]. \tag{28}$$

I show in Appendix C that $P(\pi^{(1)}, \pi^{(2)}, \eta, q) = p(\pi^{(1)}, \pi^{(2)}, \eta)\cdot q$, where $p(\pi^{(1)}, \pi^{(2)}, \eta)$ follows an elliptic PDE with natural boundary conditions.

While closed-form solutions for the PDE above as well as that for individuals' wealth processes (to be introduced) are not available, I am able to provide polynomial approximations to these PDEs using projection methods described in, for example, Judd(1992, 1999).

I can similarly solve for the wealth of agents of type 1, $X_t^{(1)}$, when they have made optimal portfolio and consumption decisions. Given their consumption choices in (20), $X_t^{(1)}$ must satisfy

$$X_t^{(1)} = \frac{1}{\xi_t^{(1)}} E^{(1)} \left[\int_t^\infty c_s^{(1)} \xi_s^{(1)} ds \right]. \tag{29}$$

Linearity of optimal consumption in output, x, implies that wealth is homogeneous of degree one in output so that $X^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta, x) = f^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta)\, x$. The PDE for $f^{(1)}(\cdot, \cdot)$ is provided in Appendix C and is solved using projection methods. Using the method of Cox and Huang(1989) the portfolio choices of agent 1 are

$$w_t^{(1)} = \sigma_{X_t}^{(1)} \cdot (\sigma_{B_t}^\top, \sigma_{P_t}^\top)^{-1} \cdot X_t^{(1)}, \tag{30}$$

where the volatilities can be obtained from the polynomial solutions of the prices. Since both bonds and stocks are in zero net supply, the portfolio choices of agents of type 2 are simply $w_t^{(2)} = -w_t^{(1)}$.

I end the characterization of the equilibrium by proving its existence and determining the constant $k = (y_1/y_2)^{\frac{1}{1-\gamma}}$, which is used in all the pricing formulas above. At the equilibrium level of k, the time-0 budget constraint of each individual in (8) is satisfied with equality. I assume the initial beliefs of each agent are at their unconditional values $\pi^{*(m)}$ given below Lemma 1 and $\eta_0 = 1$. The value of agent 1's endowment, the right-hand side of (8), can be formulated as the solution of (29) with the consumption flow rate $c_s^{(1)}$ replaced by the endowment flow $e^{(1)} x_t$ given in Assumption 5. Let us call this value $V^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta; k)$. I show that an equilibrium k is the implicit solution to the equation

$$X^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k) = V^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k), \tag{31}$$

PROPOSITION 3: *There exists an equilibrium, and for* $0 \leq \gamma < 1$, *the equilibrium is unique.*

The proof is in Appendix A.

I find the equilibrium by numerically solving for the roots of (31). While uniqueness is only established for $0 \leq \gamma < 1$, in my computations I find that the equilibrium is unique for the full range of reported γs.

III. Calibration

In this section, I describe my methodology for estimating parameters for the fundamental processes used by the two sets of agents. The method finds the parameters that jointly maximize the likelihood of each set of agents observing the fundamental processes, as well as matching the dispersion of these agents' forecasts to those available from surveys.

A. Data Series

The fundamental series that I use are the real earnings of S&P 500 companies obtained from Standard and Poor's, and the real consumption for nondurables and services obtained from the Federal Reserve Board.[12] The regime-switching model with heterogeneous beliefs is fitted to these fundamentals and a time series of dispersion in earnings forecasts.

I construct the measure of cross-sectional dispersion using data from the Survey of Professional Forecasters, available at the Federal Reserve Bank of Philadelphia. Reliable data are available from around 1970. Specifically, at each time t, let $FD_i(t,\tau)$ be the forecast of nominal profit growth τ quarters ahead by forecaster i, and $FI_i(t,\tau)$ be the forecast of the price level τ quarters ahead. I then define $FRD_i(t,\tau) = FD_i(t,\tau)/FI_i(t,\tau)$, as a measure of forecasted real earnings growth. If n_t is the number of individuals at time t, then the τ-quarters-ahead dispersion of real earnings growth at time t is

$$d(t,\tau) = \sqrt{\frac{1}{n_t - 1}\sum_{i=1}^{n_t}\left(\left(\frac{FRD_i\,(t,\tau)}{FRD\,(t)}\right) - \frac{1}{n_t}\sum_{i=1}^{n_t}\left(\frac{FRD_i\,(t,\tau)}{FRD\,(t)}\right)\right)^2}. \qquad (32)$$

The time series of the four-quarters ahead dispersion obtained in this manner is in the top panel of Figure 4.

B. Calibrated Parameter Values and Implied Beliefs

(Insert Table I about here)

The calibration methodology is described in detail in Appendix D. The parameter values used by the agents of each type are shown in Table I. The parameter values are quite compelling. The earnings drift estimates of agent 1 sandwich those of agent 2, suggesting that the former have more volatile expectations. For this reason I often refer to agent 1 as the more "volatile" investor, and agent 2 as the more "stable" investor. Agent 2 estimates that the economy shifts out of recessions more rapidly than agent 1; however, he estimates that earnings grow less rapidly in expansions. Agent 1 has an unconditional bias of -3%, about three-eights of a standard deviation from the mean of annualized earnings growth, while Agent 2 has an almost unbiased unconditional expectation.

(Insert Figure 3 about here)

In the the top panel of Figure 3 I show that the agents' beliefs are highly correlated, although in line with my comments above, agent 1's beliefs overshoot those of agent 2 in each direction. The bottom panel shows that this is also true of the agents' expected growth rate of earnings. In upturns (downturns), agent 1 (2) is relatively more optimistic. In addition, the expectations of the agents are more volatile during downturns, and tend to be more dissimilar. The reason is that both agents have higher transition probabilities out of the recession state, and their estimates are more dissimilar. Overall, the dispersion in the two agents' expected earnings growth is asymmetric and counter-cyclical.

As I mention in the introduction, the two models have very similar fits for historical earnings growth. I report the regressions of actual growth on expected growth for each of the two types of agents in Table I (bottom panel). As the table shows, the R^2s of the agents' models are 64.7% and 67.3% for agent 1 and agent 2, respectively. Both intercepts are insignificantly different from zero, and their beta coefficients are highly statistically significant and very close to one. While agent 2's model is slightly better in fitting current growth, in forecasting, agent 1's model is more accurate at horizons of one and two quarters ahead (results not shown in the table).

(Insert Figure 4 about here)

I create a model-based series of dispersions in forecasts by taking the standard deviation of forecasted growth of the two agents for any given horizon. The construction is analogous to the one from survey data in (32). The actual and model-based dispersions are shown in the top panel of Figure 4. As can be seen in the figure, both data and model-based series tend to increase in and around recessions of the U.S. economy. The two series are strongly positively correlated. A regression of the historical dispersion measure on its model-based counterpart yields an R^2 of nearly 20% (Table I, bottom panel). The beta coefficient of the regression is about 0.71, and is not significantly different from one; however, the model-based dispersion is lower on average, leading to a positive intercept term. Nonetheless, the overall chi-squared statistic for the model, which penalizes fitting errors of both agents and the dispersion error, is low with a p-value of more than 0.1.

The calibrated disagreement value process is shown in the bottom panel of Figure 4. Some comments on the dynamics of this process are made following equation (16). In addition, its path dependence (see (26)) implies that it is slow moving. This should make the disagreement value smoother than expected growth rates, as evident by comparing it with the series in the bottom panel of Figure 3. As can be seen, the disagreement value has fewer local peaks and troughs than agents' beliefs, and plays a role in generating smoother consumption series. Since this process is a martingale with a starting value of one, in a long sample it should exceed one for half the sample periods. In my sample, the process exceeds one in about 63% of the sample. Another relevant point is that even though dispersion is strongly countercyclical, the disagreement value process, being slow moving, is not. In fact the disagreement value levels during past recessions have differed quite significantly.

The two agents' equilibrium consumption levels are completely determined by the disagreement value process and realized output in the economy as seen in equation (20). I examine the calibrated consumption levels of the two types of agents in the top panel of Figure 4 (third panel). Agent 2's consumption increases in η, and thus his consumption growth rate is strongly positively correlated to positive earnings shocks (and as I show below, to stock returns) in periods when agent 2 is relatively more optimistic. Conversely, in periods when agent 1 is more optimistic, the consumption growth of agent 1 (2) will has a positive (negative) correlation with earnings shocks. The figures show that the consumption of agent 1 increases faster in and around most NBER-dated recessions and other periods of low earnings growth. Agent 2's consumption follows the opposite pattern. Both series have positive trends, and the two are negatively correlated. It is also worth noting that starting from the same level, the calibrated consumption paths cross about 12 times in the 30-year sample, although by the end of the sample, the consumption of agent 1 outpaces that of agent 2. In addition, there are fairly long swings in the domination of a type 1 agent, followed by rapid declines, that bring their average growth rates about level with type 2 agents. As I show in the next section, the volatilities of the two agents' consumption growths vary significantly over time, and on average the volatility of agent 1 is higher. However, since his mean growth rate is also higher, I find that the two agents' welfare is almost identical and the constant $k = (y_1/y_2)^{1/(1-\gamma)}$ determined in equation (31) is slightly below one in my calibrated model. I shed further light on consumption patterns when I discuss the portfolio choices and risk premia of the two types of agents in the next section.

IV. Equity Premium and Riskless Rate Results

I use the parameters of the fundamental processes calibrated in Section III to study the equity premium and related statistics implied by the model. In addition, I use different parameters for agents' preferences. I split my analysis into three parts. In Subsections IV.A and IV.B I discuss properties of the unconditional and conditional moments of the variables relevant to the puzzles, respectively, and in Appendix B I examine the effects of relaxing my assumption on stocks being in zero net supply.

A. Unconditional Moments

I use the pricing functions to generate price series and returns from the calibration exercise. For the calibrated results, I use the calibrated belief and disagreement value processes shown in Figures 3 and 4. For example, to obtain a real stock price series, I use the product of the realized proxy for dividends (50% of earnings) and the P-D ratio of $p(\pi_{1t}^{(1)}, \pi_{1t}^{(2)}, \eta_t)$, whose value is obtained from the solution of equation (C2) in Appendix C. Other variables are generated analogously. Calibrated portfolio weights held in stocks by agent 1 are shown in the bottom panel of Figure 4 (fourth panel). The weights vary between -0.2 and 0.2, as the relative optimism of agent 1 changes. As can be seen in the figure, agent 1 holds short positions more often, consistent with his relative pessimism depicted in the bottom panel of Figure 3. The weights are generally small since consumption volatility is lower than stock price volatility.

(Insert Table II about here)

The results from the calibration exercise are shown in Table II. Column (3) of the table provides the average equity volatility generated by the model. In each case average annualized stock return volatility is between 18% and 20%, and increases in γ (I discuss this effect below). This exceeds the volatility of the assumed dividend process in the model of around 7%, due to the volatility caused by the beliefs and the disagreement value. Comparing lines 2 and 12, I see that the effect of increasing the time discount from two to three increases stock volatility. The intuition is that with higher time discount, greater weight is given to current news on fundamentals.

The average model-implied riskless rate is reported in column (4). As I mention in the discussion of Proposition 2(ii), the rate is determined by three different effects. With low calibrated volatility of aggregate consumption, the precautionary savings effect is small for a very large range of investor risk aversion. In the range in which $0 < \gamma < 1$, the wealth effect is positive, while the hedging demand effect is negative. Both the wealth and hedging demand effects imply lower rates as γ increases. For example, for the sample period the average riskless rate is 2.75% and 2.17% for γ of 0.5 and 0.6, respectively (lines 1 and 2). Note that while the model's average rate is above the empirically estimated 1% average real riskless return, it is 75 to 90 basis points below the benchmark economy in which agents have homogeneous beliefs. Moreover, the standard deviation of this real rate is extremely low, between 1% and 2% at an annual rate. In the calibration, I use a time-discount factor, ρ, of 2%. Some authors have argued that a higher time discount is more reasonable. Using instead ρ of 3% raises the average rate by 1% and does not change its standard deviation (lines 2 and 12). With a higher γ, the average riskless rate drops further as agents' savings demand increases in periods of low disagreement and high dispersion. However, with a higher γ the volatility of the short rate as well as individuals' consumption volatilities increase beyond observed levels (results are not reported). For $\gamma = 0$ (log-utility) the hedging effect is zero and the riskless rate hits 4.8$\%$. For $\gamma < 0$, both effects are positive and increase as γ is lowered, pushing the riskless rate even higher. For extremely high CRRA in excess of 50, the precautionary savings effect finally dominates and the riskless rate falls again (not shown in the table).

The equity premium under the objective measure calculated as the average annualized excess returns from the model-generated price series is in column (6). For the case $\gamma = 0.5$, the equity premium is about 2.4$\%$, compared to an historical equity premium of 6.1$\%$ in this particular sample. Somewhat remarkably, when I increase γ to 0.6, the equity premium rises to 2.95$\%$. The intuition for this result is provided in the discussion of Proposition 2(iii). Essentially, the speculative component of the market prices of risk of both agent types increases in absolute value as γ increases. Less risk-averse agents take more speculative positions and their relative shares of the economy fluctuate more. For $\gamma > 0$, the speculative component of the equity premium outweighs the premium from aggregate risk in fundamentals, which declines with lower risk aversion. The cost of obtaining a higher risk premium is higher individual consumption volatilities, which I discuss below. For

$\gamma = 0.5$, the resulting Sharpe ratio generated (column 8) is about 13%, less than half its historical value. Nonetheless, at the low level of risk aversion assumed, this is a substantial improvement over current models. As γ is lowered below zero, agents speculate less, their consumptions shares become less diverse, and the speculative component of the market price of risk declines. As γ is lowered below -9, the premium from the aggregate risk in fundamentals eventually dominates and the risk premium starts increasing again, reaching 2.6% at $\gamma = -29$. This nonmonotinicity can also be viewed in Figure 1.

Using the individual consumption functions in Proposition 2(i) and the calibrated disagreement value process, I generate processes for individual consumption and calculate their sample volatilities. Of the two investors in my model, agent 1 has a higher volatility of consumption, in line with the larger volatility of his belief process. As I increase γ, the volatilities of both agents' consumption growths increase. For the case $\gamma = 0.5$, the per capita volatility is nearly 10%, within the range estimated in recent papers cited in the introduction. The higher per capita volatilities are essential for generating higher Sharpe ratios in my model relative to the homogeneous agent benchmark. However, as I show in the next subsection, this level of volatility is not in itself sufficient for generating my results. Finally, when γ is lowered below zero, individual consumption volatilities rapidly decline as agents reduce their speculation.

I can also compute the average ex-ante equity premium *expected* by agents in my model. These are shown in columns (12) and (13) for the two types of agents. The expected equity premium for each agent is calculated using the market prices of risk in Proposition 2 and the volatilities of asset prices at each date, using the calibrated belief and disagreement processes. As anticipated, I find that the equity premium of agent 1 underestimates the long-term historical equity premium in the model by 5% to 6%, while that of agent 2 is very close to its historical average. In the next subsection, where I examine conditional equity premia, I show that the "bias" of agent 1 is consistent with equation (13). However, in periods when agent 1 is more pessimistic, he is on average short in equities, and hence his strategy earns positive expected returns.

One perhaps surprising aspect of my results is that agent 1, who in the *sample* underestimates earnings growth and trades based on his model, survives. In fact, he has a slightly higher average growth rate of consumption than agent 2. This result may seem at odds with recent results by Blume and Easley(2004) and Yan(2005), who show that "irrational" agents do not survive in the long term. In their analysis, the rational agent has precise knowledge of the data generating process (DGP), whereas the "irrational" agent has a misspecified model, such as a biased estimate of the drift rate. The latter paper reports that when the bias of an agent is substantially larger than that of agent 1 in this paper, it may take hundreds of years for his wealth share of the economy to decline noticeably. Nevertheless, I note that there are key differences in my analysis in that both agents face estimation risk and neither agent knows the exact DGP. Agent 2 is "more rational" only in the sense that the econometrician's estimate of his unconditional mean equals the sample mean. The unconditional mean of the data generating process may be different from its sample mean. Moreover, both agents' models may be slightly misspecified relative to the DGP, as every model is. While I specify both agents' models without specific constraints on their rationality, I find that each model explains a similar fraction of variation of the realized earnings growth, and when trading with their models, both agents survive.

The last two columns report the equity premiums and riskless rate when beliefs are homogeneous. With a nonstochastic drift rate of aggregate output, investors' beliefs and consumption are uncorrelated and hence the component of volatility arising from belief variation does not command a market risk premium. As in David and Veronesi(2002), the equity premium equals $(1-\gamma)\cdot\sigma_q\sigma_x^\top$. These columns highlight that the equity premium absent speculation as modeled here is of the order of 0.0005$\%$, while the riskless rate is between 3% and 4% (one to two percentage points higher than the time-discount factor) for CRRA values smaller than one. Therefore, in this range, the dispersion of beliefs is the driving force in my model, and none of the other features of the calibration exercise can account for my results. For very low γ, the aggregate risk of fundamentals dominates and the risk premium for the model resembles that of the homogeneous agent economy.

A.1 Back-of-the-Envelope Approximation of Sample Moments

While calculation of the unconditional moments of asset returns and consumption as described above is cumbersome, I am able to provide some approximate calculations that are easily replicated and show that the premium in my model is close to half its empirically observed level, which is more than 50 times higher than in a benchmark economy with the same level of risk aversion and homogeneous beliefs. The exercise does miss some crucial properties of conditional moments of my model, which I discuss in Section IV.B.

I make use of the following sample moments. First, as I point out above, agent 1 underestimates earnings growth by 3% and agent 2 has an essentially unbiased estimate. Second, the average sample moment of the consumption share of agent 1 is 0.55. Third, average stock volatility in the sample is 19%.

Using these sample moments, I approximate the moments of the equity premium puzzle. First, Corollary 2 implies that the equity premium with a CRRA of 0.5 would average $0.5 \cdot 0.02 \cdot 0.19 + 0.55 \cdot \frac{0.03}{0.083} \cdot 0.19 = 0.039$, which is 1% higher than the premium reported above. Second, using (21), the approximate sample average of the riskless rate is

$$\bar{r} \simeq 0.02 - \frac{1}{2}\,(2-\gamma)\,(1-\gamma)\,0.0004 + (1-\gamma)\,0.03 - \frac{\gamma}{2(1-\gamma)} \cdot 0.55 \cdot 0.45 \cdot \frac{0.03^2}{0.083^2},$$

where $\rho = 0.02$, $\kappa^{(m)} = 0.028$, $\sigma_{x,1} = 0$, $\sigma_x \sigma_x^\top = 0.004$, $\sigma_{q,1} = 0.083$, and $k \simeq 1$. For $\gamma = 0.5$, I obtain $\bar{r} = 0.016$, which is 1% below that from the exact calculation. Similarly, from (22) and (23),

$$\bar{\phi}_q^{(1)} = 0.5 \cdot 0.02 + 0.45 \cdot \frac{0.03}{0.083} = 0.172, \quad \text{and} \quad \bar{\phi}_q^{(2)} = 0.5 \cdot 0.02 + 0.55 \cdot \frac{\bar{b}}{0.083} = 0.208.$$

By Lemma 2, for $\gamma = 0.5$, the quarterly consumption volatilities of the two agents equal these market prices of risk. Note that these volatilities are about twice as high as those reported from the exact calculation in Table II. In fact, I show in Subsection IV.B that all the moments of my model vary significantly over time and the lower unconditional consumption volatility is consistent with the reported moments of asset prices.

A.2 Bootstrapped Distributions of Asset Pricing Moments

The asset pricing results above are obtained from one 30-year sample realization of the data. A natural question to ask is whether this is a "typical" sample path. To address this question, I would like to determine standard errors of all statistics and to ascertain whether the survival of the pessimistic agent in the sample is fortuitous. Monte-Carlo simulations of the model are not informative for the analysis since the relative performance of the two agents' models must be compared with the true data generating process, which is unknown to the econometrician. I therefore resort to a standard bootstrap methodology in which "innovations" to each model are resampled with replacement and all statistics are recalculated for the case $\rho = 0.02$ and $\gamma = 0.5$. The results from 10,000 repetitions (and a basic description) of this exercise are reported in Figure 5.

(Insert Figure 5 about here)

The left panel shows the distribution of $\bar{\theta}^{(1)} - \bar{\theta}^{(2)}$, which is the relative bias of agent 1. The mean of the bias is -3.1% compared to -2.72% in my sample. The distribution is heavily negatively skewed with a minimum bias of -12%. However, 58% of the observations are above the mean value. The middle panel shows the distribution of the equity premium under the objective measure. The mean of this distribution is 2.2%, close to the mean of my sample, with a standard deviation of 0.63%. This distribution is negatively skewed, but not as much as that of the relative bias. Finally, the distribution of the riskless rate in the right panel has a mean of 2.9%, with a standard deviation of 0.2%, and it is heavily negatively skewed.

Further intuition for the relative difference in skewness of these distributions can be obtained from the shapes of the pricing functions in the top panels of Figure 2. First, as noted above, with the unconditional biases of the two types of agents, the premium of agent 2 is likely to be close to the premium under the objective measure. The right panel shows that in samples in which the relative bias of agent 1 is negative (positive), the objective premium will be positive (negative) and close to linear in this relative bias. However, when the bias is very large in either direction, the premium flattens out due to the reduction in volatility (middle panel) in periods in which agents are very confident of either state. Therefore, while the premium is likely to take on values both above and below its mean, its skewness is limited due to this last observation. The left panel shows that the riskless rate has a humped shape over the relative bias: For both negative *and* positive values of the relative bias, the riskless rate declines. This causes the distribution of the riskless rate to be more negatively skewed. Finally, note that the minimum and maximum values for the disagreement value, η_t, in the 10,000 resamplings of 30-year samples are 0.43 and 3.13, which are clearly bounded away from zero and infinity, thus implying that each agent survives with probability one.

B. Conditional Moments

Since agents in the model do not face any trading frictions, Lemma 2 implies that agents' market prices of risk and their conditional consumption volatilities are closely related. Therefore, for a given γ, statistics of individual consumption growth rates should summarize the information in agents' prices of risk. In this subsection, I take a closer look at the moments of individual consumptions that are generated by the state variables in my model, and reconcile the equity premia of agents reported in the previous subsection with conventional Euler equations.

The conditional risk premium of agent m for any given asset is the conditional covariance between his market price of risk and the asset's volatility. Equation (17) of Lemma 2 further shows that the agent m's market prices of risk equal the CRRA times the volatility of the agent's conditional consumption growth. Therefore, his risk premium is

$$\mu_P^{(m)} + \delta_t - r_t = \rho_t^{(m)} \cdot (1-\gamma) \cdot ||\sigma_{ct}^{(m)}|| \cdot ||\sigma_{Pt}||, \tag{33}$$

where $\rho_t^{(m)}$ is the conditional correlation between his consumption growth and stock returns. If the quantities on the right-hand side are time-varying (as is implied by Lemma 2), then care has to be taken to extend the implications of this pricing equation to its unconditional form (see Hansen and Richard(1987) and Campbell and Cochrane(2000)).

Several authors use the unconditional form of (33) to place bounds on the Sharpe ratios and the equity premium. If consumption and stock price volatilities are assumed constant at their expected values, then, indeed, the unconditional equity premium of investor m must be bounded by $(1-\gamma) \cdot E[||\sigma_{ct}^{(m)}||] \cdot E[||\sigma_{Pt}||]$. For example, looking at line 3 of Table II, the case in which $\rho = 0.02$ and $\gamma = 0.5$, the equity premium of agent 2 would be bounded by $0.5 \cdot (2 \cdot 0.0971) \cdot 0.1857 = 0.018$, which is clearly smaller than the ex-ante expected premium computed for agent 2 in column (13) as 2.4% (note that the consumption volatility in the table is stated in quarterly units and must be doubled to find its implications for the annualized premium). The bound is large because it assumes a correlation between the consumption growth of agent 2 and equity returns of one. If I were to further assume that the correlation is constant at its unconditional value of 0.1445 as reported in column (15), then the premium implied by the unconditional moments would be much smaller at 0.26%. This is, of course, just the correlation puzzle as stated in Cochrane(2001), who notes that the correlation between aggregate consumption and stock returns is no more than 20%, thus further deepening the equity premium puzzle. In the remainder of this section, I point out key features of the time variation of the moments of individual consumption growths and stock returns that will justify the premiums reported in Table II.

(Insert Table III about here)

I show in the remainder of this section that the conditional distribution of most calibrated variables is markedly different depending on the relative optimism of the two agents. I therefore partition all the series into two parts, with the former group having dates in which $\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$. In this group of observations, the volatile investor (agent 1) is the conditional optimist. Table III shows the conditional sample moments of various variables in my calibration exercise. It is useful also to look at the bottom panel of Figure 4, which plots time series of the conditional expectations of the dividend drift of the two types of agents. About 43% of the observations in the sample fall into the first group. The reason I partition the data in this fashion is that the sign of the conditional risk premium of agent 1 (2) is positive (negative) for observations in the first group, and the signs are reversed for observations in the alternative group.

Using the calibrated beliefs and disagreement value process, I generate the ex-ante conditional correlations, $\rho_t^{(m)} = (\phi_t^{(m)} \cdot \sigma_P^\top)/(||\phi_t^{(m)}|| \cdot ||\sigma_P)||$, and their averages in columns (1) and (2). As can be seen, the correlations of the simulated consumption growths of agents 1 and 2 are 0.94 and -0.91 in the first group, and -0.96 and 0.95 in the second group, respectively. The correlations differ from one (in absolute value) due to the presence of the small exogenous volatility of aggregate output terms in the market prices of risk. Since the correlations switch sign in the two subsamples, the unconditional correlations of the two agents are close to -0.14 and 0.14, respectively, similar to estimates in other papers. The high conditional correlations are critical in yielding conditional risk premia close to the product of individual consumption volatility and stock volatility for the two types of agents.

As columns (3) and (4) show, the average growth rate of consumption of agent 1 (2) is positive (negative) in the first group of observations, that is, consumption of each group of agents grows faster in periods when they are relatively more optimistic. Columns (5) and (6) show that both types' volatilities of consumption growth are much lower in the top group of observations: The average volatilities are about 2% in the first group and 13% in the second group. Intuition from this observation can be obtained from Figure 3, where the bottom panel shows that the difference in the expected growth rates of the two agents is small when $\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$. Column (7) shows the same for the volatility of stock returns in the two subsamples. Overall, for both types of agents, market prices of risk (consumption volatilities) are higher in absolute value in the second group of observations, in periods when the volatility of stock returns is also higher. Due to this comovement, risk premiums in the second group of observations rise (in absolute value) more than proportionally than the increase in consumption volatilities.

Within each group of observations, correlations between agents' consumption growths and stock returns are quite stable. Hence, taking expectations of equation (A1) conditional on being in either group implies that

$$
\begin{aligned}
&E\left[\bar{\mu}_{Pt}^{(m)} + \delta_t - r_t \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}\right] \\
&\quad \simeq \quad E[\rho^{(m)} \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}] \cdot E\left[||\phi_t^{(m)}|| \cdot ||\sigma_{Pt}|| \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}\right], \\
&\quad \simeq \quad E[\rho^{(m)} \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}] \cdot \{E\left[||\phi_t^{(m)}|| \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}\right] \cdot E\left[||\sigma_{Pt}|| \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}\right] \\
&\quad + \quad \text{Cov}\left[||\phi_t^{(m)}||, ||\sigma_{Pt}|| \,|\, \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}\right\}. \qquad (34)
\end{aligned}
$$

The first component of (34) provides the risk premium due to higher average volatilities of consumption growth and stock prices for a given group. The second component measures the risk premium due to changes in dispersion of beliefs within the group of observations, which causes comovement of the two volatilities. Below, I look at the sample conditional moments in this equation to decompose the risk premium.

The two components of the ex-ante risk premiums are given in columns (8) and (10) for agent 1 and (9) and (11) for agent 2. Note that by Lemma 2, the norm of the market prices of risk must equal the product of the agents' CRRA and the norm of their consumption volatilities. Indeed, by comparing the norms of the consumption volatilities in (5) and (6), I find that the first component of each agent (columns (8) and (9)) is very close to that implied by Lemma 2. Due to low volatilities in the first group of observations, these components are very small, on the order of 0.3%. In the second group of observations this component is much larger at -3.4% and 2.5% for the two agents. The second components, shown in columns (10) and (11), capture the portion of the risk premia of the two agents arising from the covariance of the market prices of risk and stock volatility within each of the subgroups. These components are also much larger in absolute value in the second group of observations, and are nearly as large as the first components. Overall, the conditioning shows that about half the generated premium arises from higher average consumption volatilities in periods of higher average stock market volatility. The other half is generated from the covariance between these quantities within each subgroup. Due to the sign of the correlations of the two types of agents, the premia of the volatile (type 1) agents are positive in the first group of observations, when they are conditional optimists, negative in the second group, and negative overall. The reverse holds for stable (type 2) agents. The sum of the components, as shown in columns (12) and (13), is again small in the first group of observations, and equals -5.3% and 5.2% for the second group. The unconditional expectations of these premia equal -2.8% and 2.59% as in Table II. I come back to this issue after discussing the conditional dispersions of agents' expected growth rates.

Using the calibrated price process, I next report the ex-post realized excess returns of stocks in column (14). Somewhat surprisingly, the excess returns are nearly 4% in the first group of observations and much smaller at about 1% in the second group. The unconditional average of the excess returns is about 2.4%, as reported in Table II. As I note earlier, given the small positive bias of agent 2 in estimating earnings growth, his ex-ante risk premium should be fairly close to the risk premium under the objective measure. For the full sample as reported in the previous subsection, I indeed find that these two premia are quite similar. Therefore, we can infer that the equity premium under the objective measure is close to -0.8% and 5.2% in the two groups of data as reported in column (13). The relative size of these premia varies inversely with realized excess returns on stocks reported in column (14), which means that stock prices fall in periods of higher ex-ante forward-looking risk premia under the objective measure. Indeed, the excess returns over the following one to four quarters are large and positive in the second group, and are negative in the first group of observations (results not shown in the table).

Next, I discuss the impact of dispersion in beliefs on both the riskless rate and the cross-sectional dispersion in consumption growth in the two groups of data. As can be seen in column (15), the riskless rate is quite high at nearly 3.56% in the first group, and 2.13% in the second. Since the two types of agents disagree more on the transition probability from the weak to strong dividend state (Table I), their beliefs are more volatile in the second set of observations, which earlier I note are periods of weaker earnings growth. Indeed, as column (16) shows, the difference in expected growth rates is much larger in the second group of observations, in periods when volatile (type 1) agents are relatively more pessimistic. Columns (17) and (18) indicate that both the historical dispersion and the model-fitted dispersion are about 25% higher in the second group relative to the first group of observations. Further, as the left panel of Figure 2 shows, the riskless rate in the economy falls in periods of higher dispersion due to the higher speculative opportunities in such periods (see the discussion following Proposition 2). Therefore, periods of weak growth in fundamentals are also accompanied by lower (real) rates.

I now address the issue of whether the difference in the two agents' ex-ante risk premia is consistent with their expected growth rates of fundamentals. The relationship between these variables can be approximated from equation (14). Focusing on the second group of observations, the average difference in expected dividend growth rates is about -5%, and the volatility of stock returns is 23.7%, implying a difference in equity premia of around -14% (roughly $-1/0.083 \cdot 0.237 \cdot 0.05$), which is larger in absolute value than the -10.5% difference in the ex-ante premia reported in columns (12) and (13) from the exact calculation. In the first group of observations, the difference in equity premia is positive and smaller in magnitude, consistent with the smaller difference in estimated fundamentals' growth rates.

The higher volatilities of consumption and dispersion in beliefs also lead to more dispersion in growth rates of consumption across agents. As column (19) reports, the unconditional mean of the cross-sectional standard deviation (cs-sd) is 0.077 at a quarterly rate and it is about twice as high in the second group of observations, which are periods of low realized returns and high volatility. The negative relationship between the cs-sd in consumption growth and stock returns is consistent with the analysis in Constantinides and Duffie(1996), and is endogenously generated in equilibrium. However, unlike their model, the endogenous shocks to consumption in my model are not permanent, and hence they do not cause a trend increase in the cs-sd with the age of agents. Estimates of the cs-sd depend critically on the filtering method used in empirical studies. Jacobs and Wang(2004) provide an estimate of 0.1 based on age- and education-based cohorts in the Consumer Expenditure Survey (CEX) data, while Storesletten, Telmer, and Yaron (2004) report that it varies between 0.06 and 0.105 (both estimates at a quarterly rate) among cohorts of individuals in the Panel Study of Income Dynamics data. This latter study finds that the average cs-sd increases by 75% as the macroeconomy moves from peak to trough. My model-based average cs-sd lies within the range of these estimates, and even though my conditioning criterion is somewhat different, displays countercyclical variation as in Storesletten, Telmer, and Yaron (2004).

Conclusions

I show that in a model in which agents have heterogeneous beliefs about the state of fundamental growth, the risk premium increases with lower risk aversion (as long the CRRA is less than one) because the exposure to speculative risk is endogenous, increasing as less risk-averse agents undertake more aggressive trading strategies. The model implies that the equity premium is higher, the riskless rate is lower, and consumption growth is more volatile and dispersed in periods when the dispersion among agents' expectations of future growth is high and when the disagreement between their models is low. My calibrated model shows that these conditions are more likely to prevail during periods of weak fundamental growth. These stylized facts have some support in recent empirical research on individual-level consumption Storesletten, Telmer, and Yaron (2004).

The degree of freedom I take relative to existing work on the equity premium is the higher level of per capita consumption volatility. The higher volatility in itself justifies only one-tenth of the premium in my model, but its positive covariance with stock volatility, and the high conditional correlations between individuals' consumption growth and stock returns, which switch signs depending on the relative optimism of agents, together raise the model's unconditional equity risk premium to nearly half its historical level. The former property implies that higher risk (stock volatility) is also priced higher, while the latter helps address the consumption correlation puzzle by generating high conditional correlations, but unconditional correlations near empirical estimates. While exact properties of individual consumption processes are hard to measure and verify, my results support a growing literature that finds useful information for asset pricing in statistics of cross-sectional consumption growth (see, for example, Brav, Constantinides, and Geczy (2002), Jacobs and Wang(2004)). I view my results as complementary to this line of research, providing further empirical predictions to be tested at the individual level.

Notes

[1]I borrow a colorful description of this kind of risk from Abreu and Brunnermeier (2003):

> For example, when Stanley Druckenmiller, who managed George Soros's $8.2 million Quantum Fund, was asked why he didn't get out of internet stocks earlier even though he knew that technology stocks were overvalued, he replied that he thought the party wasn't going to end so quickly. In his words "We thought it was the eighth inning, and it was the ninth."

While dispersion of beliefs is a key ingredient of the "bubble" equilibrium in Abreu and Brunnermeier (2003), they do not analyze its impact on the risk premium.

[2] In this sense, even though the number of traded long-lived securities equals the number of shocks in the economy, so that markets are complete, they are effectively incomplete.

[3] Among more recent work, Shefrin(2001), Scheinkman and Xiong(2003), Anderson, Ghysels, and Juergens (2005), Dumas, Kurshev, and Uppal(2005), and Kogan, Wang, Ross, and Westerfield(2006) study asset pricing implications of heterogeneous beliefs models.

[4] This results because I have a pure exchange economy with no explicit production process or fruit-bearing tree as in Lucas(1978). The Euler equation for asset prices is the same as that in an economy in which the stock is in positive net supply, although the market prices of risk will in general differ from the latter case. Note that in Mehra and Prescott(1985) the ratio of aggregate dividends to consumption is identically one in each period, while in my model it is identically zero. As evident from the analysis in Cecchetti, Lam, and Mark(1990) and Brennan and Xia(2001), the equity premium puzzle is robust to the assumption of stocks being in zero net supply. I show later that retaining all my assumptions, and in addition assuming that agents have homogeneous beliefs, my model equity premium is very small as well. Empirically, stocks in the U.S. are in "small" positive net supply: For instance, Cecchetti, Lam, and Mark(1990) note that the dividend-consumption ratio is of the order of 2% to 5%. In Appendix B, I show by generalizing my model that the market prices of risk and riskless rates with such small supply of stock are very similar to the case modeled here.

[5] Lemma 1 in Huang and Pagès(1992) shows that the measures are mutually singular if $\int_0^\infty \sigma_{\eta s} = \infty$, almost surely as is true if $\hat{\nu}^{(1)} \neq \hat{\nu}^{(2)}$. As a simple comparison, the

objective and risk-neutral measures in the infinite horizon problems in Samuelson(1965) and Merton(1990) are mutually singular.

[6] The condition can also be written as $c_t^{(m)} = I_m(y_m \xi_t^{(m)})$, where $I_m(z)$ is the inverse of $u'_m(c^{(m)})$, and y_m is the Lagrange multiplier with respect to the budget constraint. By Assumption 4, the marginal felicity function is monotonically declining and satisfies the Inada conditions implying a unique solution for $I_m(.)$. The existence of an optimal solution for the complete variational problem including consumption and portfolio choices can be established by verifying a boundedness condition on the felicity function, and Lipschitz and growth conditions on the SPD functions with respect to each state variable (for the finite and infinite horizon cases, respectively, see Cox and Huang(1991) and Huang and Pagès(1992)). For $0 \leq \gamma < 1$ it is straightforward to verify these conditions from the explicit functional forms for r and $\phi^{(m)}$ in Proposition 2 below (see Footnote 9).

[7]I assume that the transversality condition $\lim_{t \to \infty} E^{(m)}[\exp(-\int_0^t r_s ds) P_{it}] = 0$ holds for each agent, so that only fundamental valuations are compatible with equilibrium.

[8] This effect has been well known since the papers of Hakansson(1971) and Merton(1973) for the case of general state variables and homogeneous beliefs. My incremental contribution is to study the effect of simultaneous changes in the opportunity sets of two types of agents, creating buying opportunities for one type of agent and selling opportunities for the other type.

[9] In addition, the derivative of the function $\left(k\, \eta_t^{\frac{1}{1-\gamma}}\right) / \left(1 + k\, \eta_t^{\frac{1}{1-\gamma}}\right)^p$, for $p = 1, 2$, with respect to η is bounded on $(0, \infty)$ for $0 \leq \gamma < 1$, and hence the riskless rate and market prices of risk satisfy the sufficient Lipschitz conditions for the existence of investors' variational problem in footnote 6.

[10]While agents trade on their own models as in models of investor overconfidence (for example, Daniel, Hirshleifer, and Subrahmanyam(2001)), their market price of dividend risk increases when their model predictions are more optimistic relative to traders of the other type.

[11]Using the same steps I can show that the statement of the corollary holds when there is an arbitrary number of types. Once again the premium under the objective measure will remain large if the consumption shares of the pessimists remain large enough after trading.

[12]Following Longstaff and Piazzesi(2003), I assume that "true" dividends are one-half of earnings, an assumption that matches the average historical dividend yield but has twice its volatility. I might add that the average historical equity premium calculated for my sample with this approximation is very close to the average premium calculated with historical dividends. This assumption on the payout series enables me to match the historical volatility of stock returns, whereas the pure dividend payout assumption does not.

Table I
Two-State Heterogeneity Model Calibration

Top Panel: GMM estimates of the following (discretized) model for real consumption, x_t, and real earnings, q_t:

$$x_{t+1} = x_t \cdot e^{(\kappa_t^{(m)} - \frac{1}{2}\sigma_x \sigma_x')\Delta t + \sigma_x \varepsilon_{t+1}} \; ; q_{t+1} = q_t \cdot e^{(\theta_t^{(m)} - \frac{1}{2}\sigma_q \sigma_q')\Delta t + \sigma_q \varepsilon_{t+1}}.$$

where $\sigma_q = (\sigma_{q1}, \sigma_{q2})$, $\sigma_x = (0, \sigma_{x,2})$, and $(\theta_t^{(m)}, \kappa_t^{(m)})$ jointly follows a two-state regime-switching model. Note that the variance-covariance matrix between the two fundamental shocks is identified by setting $\sigma_{x1} = 0$. I estimate the quarterly transition probability matrix with estimates and standard errors as shown. The implied generator is $\Lambda^{(m)} = \sum_{i=1}^{\infty} (-1)^{i+1} \cdot \left((P^{(m)}(0.25))^4 - I\right)^i / i$, whose value I estimate using a series approximation of length 10 (see Israel, Rosenthal, and Wei(2001)). The GMM errors include the scores of the likelihood function of each type of agent and the difference in model-implied and historical dispersion in forecasts of Professional Forecasters as described in Appendix D. The $\chi^2(4)$ statistic for the specification test of the model is 7.6341, which has a *p*-value of 0.1059. Bottom Panel: Standard errors of parameter estimates are in parentheses. Units of measurement are quarterly and in percentage points. *T*-statistics are in parentheses. All *t*-statistics are adjusted for heteroskedasticity and autocorrelation using the methodology of Newey and West(1987). Figure 3 (top panel) shows the belief processes of the two agents. The top panel of Figure 4 shows the actual and model-implied four-quarter-ahead dispersions of earnings growth, which are in the third regression.

Series Used: Real Earnings, Real Consumption,
and Dispersion of Earnings Growth Forecasts
Time Span (Quarterly): 1971-2001

	Agent 1				Agent 2			
Drifts:	$\theta_1^{(1)}$	$\theta_2^{(1)}$	$\kappa_1^{(1)}$	$\kappa_2^{(2)}$	$\theta_1^{(2)}$	$\theta_2^{(2)}$	$\kappa_1^{(2)}$	$\kappa_2^{(2)}$
	-0.2440 (0.0194)	0.0828 (0.0289)	0.0280 (0.0103)	0.0280 (0.0103)	-0.2305 (0.0192)	0.0795 (0.0258)	0.0280 (0.0103)	0.0280 (0.0103)
Generator Elements:	$\lambda_{12}^{(1)}$	$\lambda_{21}^{(1)}$	${P_{12}}^{(1)}$	${P_{21}}^{(1)}$	$\lambda_{12}^{(2)}$	$\lambda_{21}^{(2)}$	${P_{12}}^{(2)}$	${P_{21}}^{(2)}$
	0.5061	0.3427	0.1611 (0.0612)	0.0772 (0.0444)	1.3194	0.3727	0.2749 (0.0656)	0.0776 (0.0462)
Volatilities:	$\sigma_{q,1}$	$\sigma_{x,1}$	$\sigma_{x,2}$					
	0.0833 (0.0003)	0.0109 (0.0001)	0.0200 (0.0001)					

Model Fits: $\Delta \log(q)\,(t) = \alpha + \beta \cdot (\theta_1^{(m)} \pi_1^{(m)}(t|t) + \theta_2^{(m)} \pi_1^{(m)}(t|t)) + \epsilon(t)$, *m* = 1,2

Agent 1			Agent 2		
$\hat{\alpha}$	$\hat{\beta}$	R^2	$\hat{\alpha}$	$\hat{\beta}$	R^2
0.0932	1.4885	0.6476	-0.2116	1.8240	0.6737
(0.2224)	(8.8857)		(-0.5248)	(10.0718)	

Dispersion: $d(t,4) = \alpha + \beta \cdot d(\pi^{(1)}(t,4), \pi^{(2)}(t,4)) + \epsilon(t)$

$\hat{\alpha}$	$\hat{\beta}$	R^2
4.1301	0.7190	0.1982
(12.1873)	(4.0921)	

Table II

Calibrated Equity Premium and Related Statistics (1971 to 2001)

The unconditional means of the following variables (annualized units, unless stated) are generated from the model: col. (3), $||\sigma_P||$, volatility of equity returns, col. (4), μ_R, riskless rate, col. (5), σ_R, volatility of riskless rate, col. (6), $\mu_P - \mu_R$, equity premium (objective measure), col. (7), $(\mu_P - \mu_R)/||\sigma_P||$, Sharpe ratio, col. (8) and col. (9), σ_{c_j}, $j = 1, 2$, volatilities (quarterly rate) of individual consumption growth, col. (10), $\bar{\sigma_c}$, per capita volatility (quarterly rate) of consumption growth, cols. (11) and (12), $\mu_E^{(j)} - \mu_R$, $j = 1, 2$, equity premium under measure of two different agents, cols. (13) and (14), $\rho^{(m)}$, correlations between consumption growth and stock returns for the two agent types, and cols. (15) and (16), $\mu_E^{ho} - \mu_R^{ho}$, and μ_R^{ho}, equity premium and riskless return under homogeneous beliefs. Equity premia and all other statistics are calculated using the solutions to the PDEs in Section II. A, using the parameter values in Table I. The belief process, $\{\pi_t^{(m)}\}$, is generated using the discretized version of the SDE in Lemma 1 as shown in (D1) and (D2) of Appendix D. The disagreement value process is generated using the discretized version of (26). Calibrated belief and disagreement value processes are shown in Figures 3 and 4. The price-dividend ratio at period t is $p(\pi_t^{(1)}, \pi_t^{(2)}, \eta_t)$ and is obtained from the solution of PDE (C2) in Appendix C. The calibrated price at time t is given by $p(\pi_t^{(1)}, \pi_t^{(2)}, \eta_t) \cdot q_t/2$, where q_t are S&P 500 earnings per share. Other variables are similarly calculated. The riskless rate is calculated from (21) as shown in Figure 2. Agents' consumptions are in (20) and are shown in the third panel of Figure 4. The ex-ante expected premium of each agent is obtained from equation (A1). The risk premium for the homogeneous investor case is $(1-\gamma)\sigma_x\sigma_q'$, and the riskless rate is in (21) with the last term set to zero.

	(1) ρ	(2) γ	(3) $\|\|\sigma_P\|\|$	(4) μ_R	(5) σ_R	(6) $\mu_P - \mu_R$	(7) $\frac{\mu_P-\mu_R}{\|\|\sigma_P\|\|}$	(8) $\|\|\sigma_{c_1}\|\|$	(9) $\|\|\sigma_{c_2}\|\|$	(10) $\|\|\bar{\sigma}_c\|\|$	(11) $\mu_P^{(1)} - \mu_R$	(12) $\mu_P^{(2)} - \mu_R$	(13) $\rho^{(1)}$	(14) $\rho^{(2)}$	(15) $\mu_P^{ho} - \mu_R^{ho}$	(16) $\mu_{R^{ho}}$
1	0.02	0.6	0.192	0.021	0.017	0.029	0.153	0.138	0.124	0.131	-0.035	0.032	-0.140	0.145	3×10^{-4}	0.031
2	0.02	0.5	0.186	0.028	0.011	0.024	0.130	0.102	0.097	0.099	-0.028	0.026	-0.139	0.144	4×10^{-4}	0.034
3	0.02	0.4	0.183	0.032	0.008	0.020	0.109	0.087	0.081	0.084	-0.024	0.022	-0.139	0.144	4×10^{-4}	0.036
4	0.02	0.3	0.181	0.036	0.005	0.018	0.094	0.074	0.068	0.071	-0.021	0.018	-0.139	0.144	5×10^{-4}	0.039
5	0.02	0.0	0.165	0.048	0.000	0.016	0.109	0.051	0.049	0.050	-0.019	0.018	-0.111	0.124	8×10^{-4}	0.048
6	0.02	-4.0	0.144	0.162	0.022	0.014	0.118	0.024	0.023	0.023	-0.015	0.017	-0.109	0.114	0.004	0.154
7	0.02	-9.0	0.133	0.287	0.024	0.013	0.120	0.012	0.011	0.012	-0.008	0.017	-0.105	0.108	0.008	0.278
8	0.02	-14.0	0.125	0.402	0.026	0.017	0.160	0.011	0.010	0.010	0.003	0.021	-0.103	0.106	0.012	0.393
9	0.02	-19.0	0.119	0.507	0.026	0.019	0.184	0.010	0.010	0.010	0.010	0.023	-0.101	0.106	0.017	0.498
10	0.02	-24.0	0.114	0.602	0.027	0.023	0.201	0.010	0.010	0.010	0.015	0.025	-0.100	0.104	0.021	0.593
11	0.02	-29.0	0.111	0.688	0.027	0.026	0.234	0.010	0.010	0.010	0.018	0.026	-0.100	0.103	0.025	0.679
12	0.03	0.5	0.188	0.037	0.011	0.024	0.129	0.102	0.097	0.099	-0.028	0.026	-0.139	0.144	4×10^{-4}	0.043

Table III
Conditional Moments of Model-Generated Variables

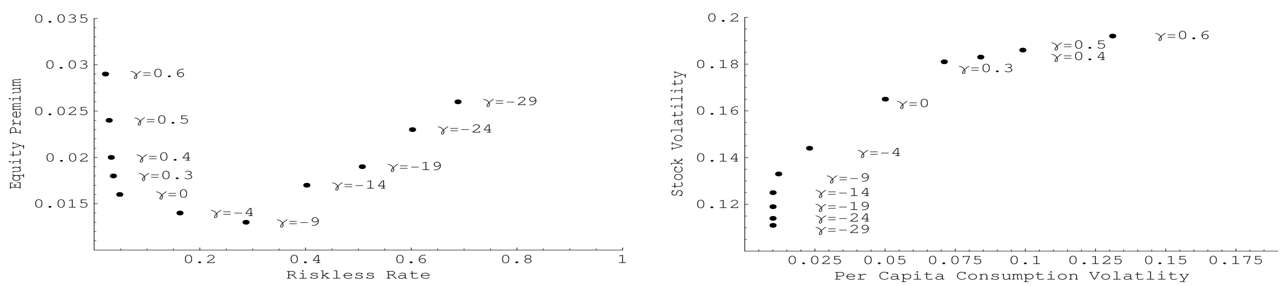
The following (annualized, unless stated) conditional moments are generated from the model for the case when $\rho = 0.02$ and $\gamma = 0.5$ corresponding to line 3 of Table II. Cols. (1) and (2), $\rho^{(m)}$, *m*=1,2, are the conditional correlations of ex-ante consumption growth with stock returns of the two types of agents respectively, which at time t is calculated as $(\phi_t^{(m)} \cdot \sigma_P^\top)/(||\phi_t^{(m)}|| \cdot ||\sigma_P)||$, where $\phi^{(m)}$ are the market prices of risk in Proposition 2 (iii). Cols. (3) and (4), $\mu_c^{(m)}$, are individual ex-post mean consumption growth rates (quarterly), and cols. (5) and (6), $||\sigma_c^{(m)}||$, are individual ex-post consumption growth rate volatilities (quarterly). These are calculated using the agents' consumption functions in (20), and the consumption paths are shown in the third panel of Figure 4. The calibrated disagreement value process is in Figure 4. Col. (7) is the volatility of equity returns obtained from the solutions to the PDEs in Section II. A as shown in equation (C6) of Appendix C using the parameter values in Table I. For its generation, in addition to the disagreement value process, I use the calibrated beliefs of each agent shown in Figure 3. Cols. (8) and (9), $\rho^{(m)} E[||\phi^{(m)}||] \cdot ||\sigma_P||$, are the products of the conditional correlations in cols. (1) and (2) and expectations of the norms of the market prices of risk and stock volatility of the two types of agents respectively. Cols. (10) and (11), $\rho^{(m)}\text{Cov}[||\phi^{(m)}||, ||\sigma_P||]$, are the products of the conditional correlations in cols. (1) and (2) and covariances of the norms of the market prices of risk and stock volatility of the two types of agents respectively, and cols. (12) and (13), $\mu_E^{(m)} - \mu_R$, are the ex-ante expected equity premia of the two agents, respectively, calculated as in equation (A1). Col. (14), $\mu_P - \mu_R$, is the ex-post realized cum-dividend realized excess return on equities calculated from the calibrated price and dividend process described in the footnote to Table II. Col. (15), μ_R, is the riskless return calculated as in equation (21). Col. (16), $E[\bar{\theta}_1^{(1)} - \bar{\theta}_t^{(2)}]$, is the difference in expectations of earnings growth of the two types of agents. Col. (17), $d(t, 4)$, is the four-quarter-ahead dispersion of Professional Forecasters, and col. (18), $\hat{d}(t, 4)$, is its model-fitted (scaled) counterpart (the two series are shown in Figure 4). Col. (19), σ_{cs}, is the cross-sectional standard deviation of ex-post consumption growth rates (quarterly) across agents.

Cases	Prob.	(1) $\rho^{(1)}$	(2) $\rho^{(2)}$	(3) $\mu_c^{(1)}$	(4) $\mu_c^{(2)}$	(5) $\|\|\sigma_c^{(1)}\|\|$	(6) $\|\|\sigma_c^{(2)}\|\|$	(7) $\|\|\sigma_P\|\|$
$\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$	0.4308	0.9431	-0.9127	0.0255	-0.0098	0.0416	0.0769	0.1184
$\bar{\theta}_t^{(1)} < \bar{\theta}_t^{(2)}$	0.5691	-0.9596	0.9448	-0.0038	0.0152	0.1482	0.1124	0.2366
Averages		-0.1398	0.1445	0.0088	0.0044	0.1023	0.0971	0.1857

Cases	Prob.	(8) $\rho^{(1)} E[\|\|\phi^{(1)}\|\|] \cdot E[\|\|\sigma_P\|\|]$	(9) $\rho^{(2)} E[\|\|\phi^{(2)}\|\|] \cdot E[\|\|\sigma_P\|\|]$	(10) $\rho^{(1)}\text{Cov}[\|\|\phi^{(1)}\|\|, \|\|\sigma_P\|\|]$	(11) $\rho^{(2)}\text{Cov}[\|\|\phi^{(2)}\|\|, \|\|\sigma_P\|\|]$	(12) $\mu_E^{(1)} - \mu_R$	(13) $\mu_E^{(2)} - \mu_R$
$\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$	0.4308	0.0046	-0.0083	-0.0003	0.0003	0.0043	-0.0086
$\bar{\theta}_t^{(1)} < \bar{\theta}_t^{(2)}$	0.5691	-0.0336	0.0251	-0.0195	0.0269	-0.0531	0.0520
Averages		-0.0171	0.0107	-0.0112	0.0152	-0.0284	0.0259

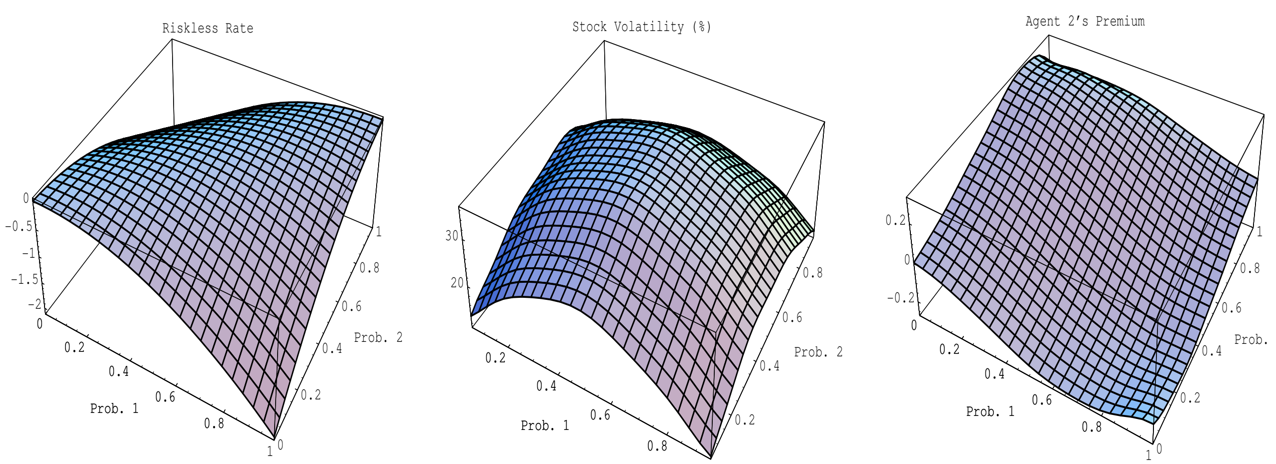
Cases	Prob.	(14) $\mu_P - \mu_R$	(15) μ_R	(16) $E[\bar{\theta}_1^{(1)} - \bar{\theta}_t^{(2)}]$	(17) $d(t, 4)$	(18) $\hat{d}(t, 4)$	(19) σ_{cs}
$\bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}$	0.4308	0.0422	0.0356	0.0048	6.3881	6.0375	0.0492
$\bar{\theta}_t^{(1)} < \bar{\theta}_t^{(2)}$	0.5691	0.0104	0.0213	-0.0514	7.6055	7.8357	0.0984
Averages		0.0241	0.0275	-0.0272	7.0603	7.0604	0.0771

Figure 1. The effects of risk aversion on the average riskless rate, equity premium, per capita consumption volatility, and stock volatility.



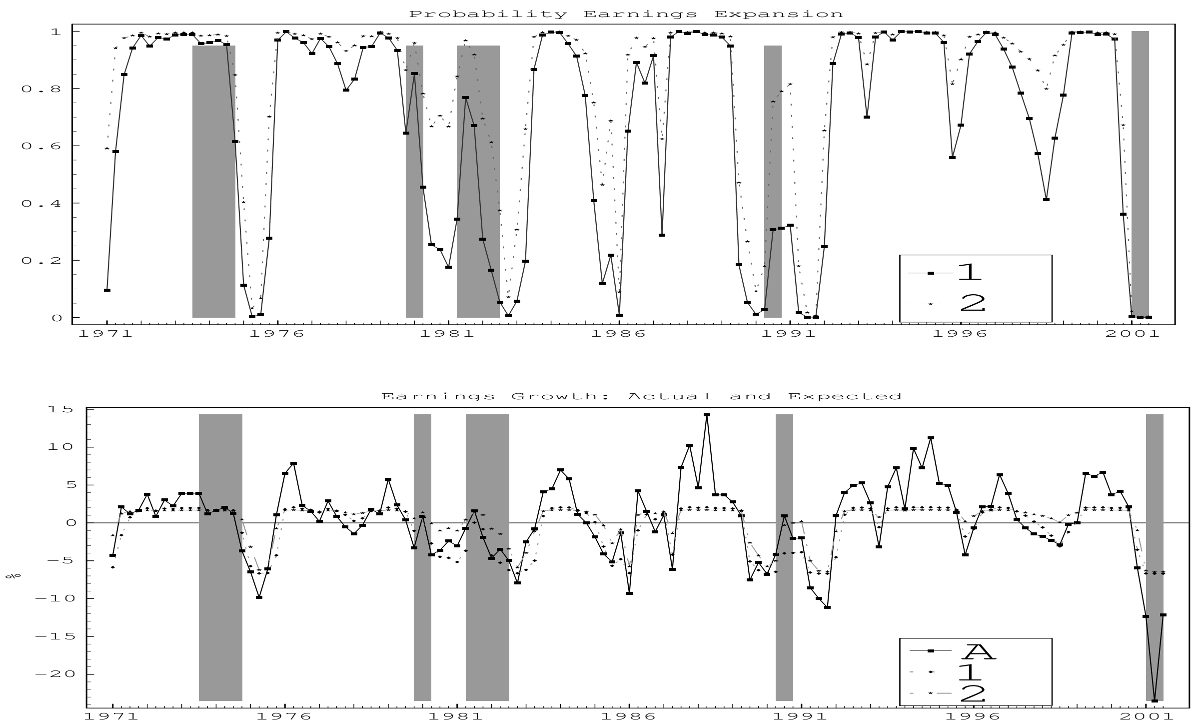
The left panel plots the riskfree rate and the equity premium for alternative levels of γ, where agents' constant coefficient of relative risk aversion is $1-\gamma$. The right panel plots per-capita consumption volatility and stock volatility for alternative levels of γ. Details of the calculation are provided in the footnote to Table II.

Figure 2. The riskless rate, stock return volatility, and agent 2's equity premium.



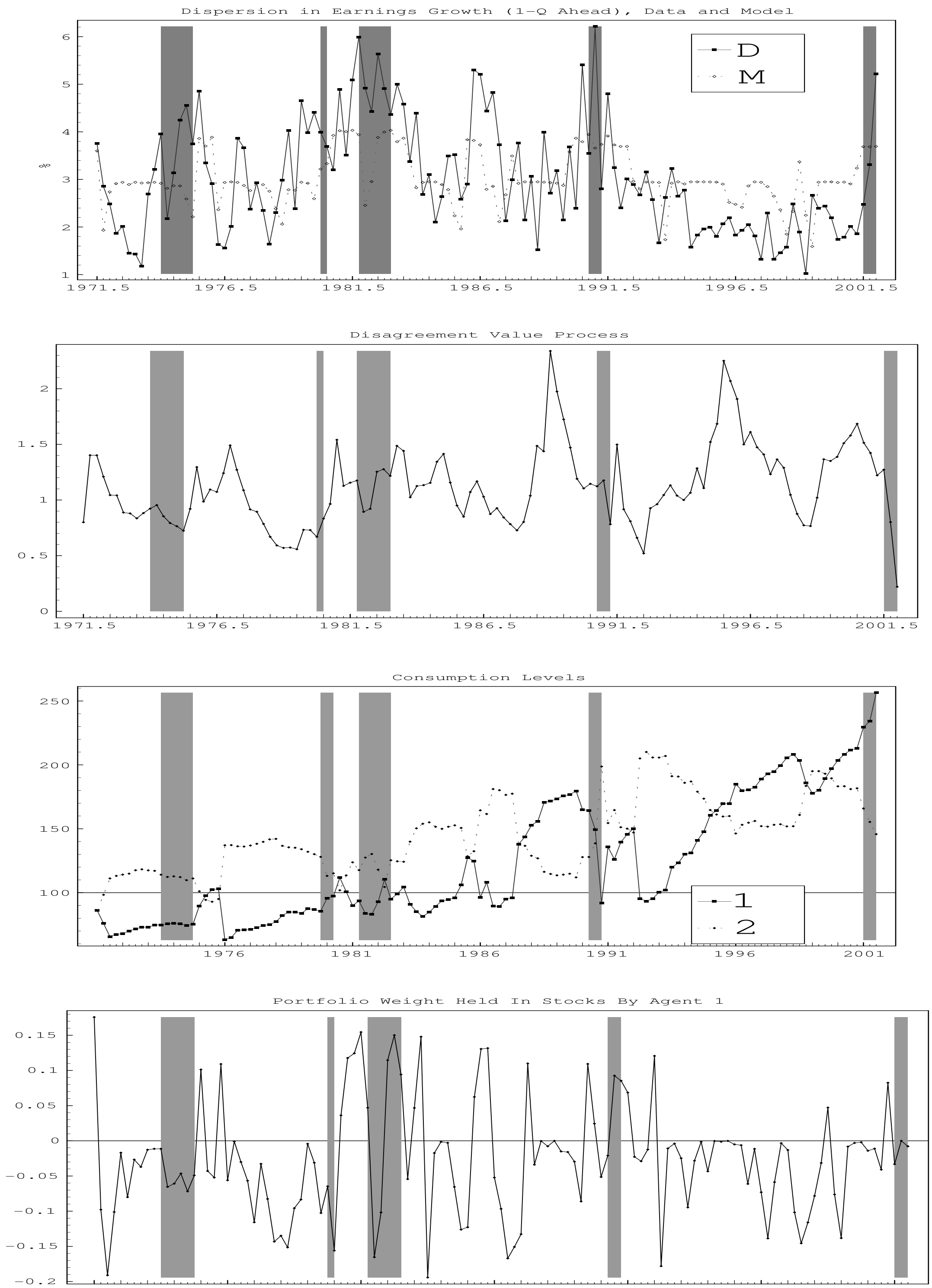
The riskless rate and stock return volatility are in equations (21) and (C6). The risk premium of agent 2 is $-\phi^{(2)}\sigma_p^\top$, where the market prices of risk are in equations (23) and (25). In the figures I set the disagreement value to one, and Prob 1. and Prob 2. are the two agents' beliefs of earnings currently being in an expansion state. The parameters of the fundamental processes are shown in Table I. In addition, I use $\rho = 0.02$ and $\gamma = 0.5$.

Figure 3. Investor beliefs, expected growth rates of earnings



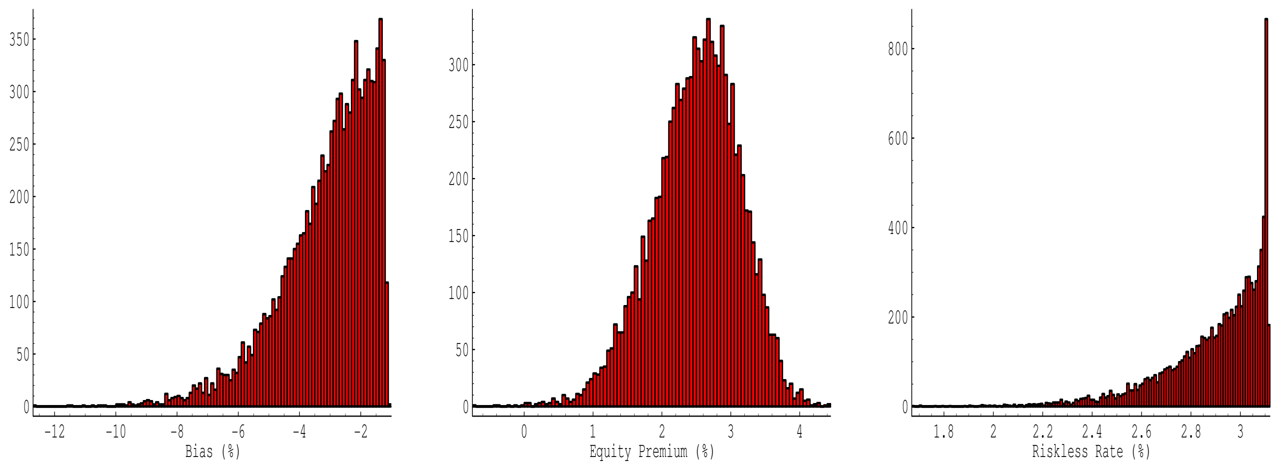
The top panel has the time series of filtered beliefs about real earnings growth of the two types of agents. Filtered beliefs of the two agents are obtained from the discretized version of the belief processes in Lemma 1 as shown in equations (D1) and (D2) of Appendix D using the calibrated parameters for each type of agent shown in Table I. The bottom panel displays the actual and expected earnings growth of the two types of agents using these filtered beliefs.

Figure 4. Dispersion in earnings growth, the disagreement value process, and agents' decisions



The first panel displays the model-implied dispersion and the dispersion of four-quarter-ahead earnings growth from the Survey of Professional Forecasters calculated as shown in (32). Model series are analogously calculated. The second panel shows the disagreement value process, $\{\eta_t\}$, implied by the belief series shown in Figure 3 and equation (16). The third panel shows the consumption levels of each agent as calculated from (20) using the disagreement value process, $\{\eta_t\}$, in the second panel. The fourth panel shows the proportion of wealth in the stock held by investors of type 1 (pessimistic investors) calculated as $w_t^{(1)} = \sigma_{X_t}^{(1)} \cdot (\sigma_{B_t}^\top, \sigma_{P_t}^\top)^{-1}$. The volatilities of stocks, bonds, and wealth are calculated as described in Section II. A and at time t are calculated conditional on agents' beliefs and disagreement value. Parameter values for fundamentals are as shown in Table I, and in addition, $\rho = 0.02$ and $\gamma = 0.5$.

Figure 5. Bootstrapped distributions of the relative bias of agent 1, the equity premium, and the riskless rate.



Distributions of the three statistics are obtained by resampling the series of fitted *innovations*, $\{\tilde{W}(t)\}$, of type m agents, with replacement $J = 10,000$ times (see, for example, Runkle(1987)). The innovations at time t are $\tilde{W}^{(m)}(t) = (\Sigma^\top)^{-1}[\Delta \log(y)(t) - (\bar{\nu}^{(m)}(t|t-\Delta t) - \frac{1}{2}\,\text{diag}(\Sigma\,\Sigma^\top))\,\Delta t]$, where $\bar{\nu}^{(m)}(t|t-\Delta t)$ is type *m*'s expected value of the drift conditional on information available up to time $t-\Delta t$. Paths of fundamentals on the jth resampling are created using $\log(y)(t,j) = \log(y)(t-\Delta t, j) + \bar{\nu}^{(m)}(t|t-\Delta t, j) - \frac{1}{2}\,\text{diag}(\Sigma\,\Sigma^\top)\,\Delta t + \Sigma\,\tilde{W}^{(m)}(t,j)$. Updating for each type of agent's beliefs and the disagreement value on the following discrete interval is as described in Appendix D. Parameter values for fundamentals are as shown in Table I, and in addition, $\rho = 0.02$ and $\gamma = 0.5$. Statistics along each path are computed as described in the footnote to Table II.

Appendix A: Proofs

Proof of Corollary 1: To establish that ϱ_t (a zero drift process) is a martingale on $[0,t]$ under $\mathcal{P}^{(1)}$, it is sufficient to show that the Novikov condition holds, that is, $E^{(1)} \exp[1/2 \int_0^t (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})^\top (\Sigma\Sigma^\top)^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) ds] < \infty$ (see, for example, Proposition 2.24 in Nielsen(1999)). But

$$
\begin{aligned}
E^{(1)} \exp & \left[1/2 \int_0^t (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})^\top (\Sigma\Sigma^\top)^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) ds \right] \\
& < \exp \left[1/2 (\bar{\nu} - \underline{\nu})^\top (\Sigma\Sigma^\top)^{-1} (\bar{\nu} - \underline{\nu}) \cdot t \right] < \infty,
\end{aligned}
$$

where $\bar{\nu} = \max_i \nu_i^\top (\Sigma\Sigma^\top)^{-1} \nu_i$ and, $\underline{\nu} = \min_i \nu_i^\top (\Sigma\Sigma^\top)^{-1} \nu_i$. Since ϱ_t is positive and finite for all t, the measure $\mathcal{P}_t^{(2)}(A) = E^{(1)}[1_A \cdot \varrho_t]$ is equivalent to $\mathcal{P}_t^{(1)}$. An application of Girsanov's Theorem to the relation between the two innovation processes in (7) then implies that the Radon-Nikodym derivative of $\mathcal{P}_t^{(2)}$ with respect to $\mathcal{P}_t^{(1)}$ is ϱ_t. ■

In proving Proposition 1, the following result is useful.

LEMMA 3: If $(\phi_t^{(1)} - \phi_t^{(2)})^\top = \Sigma^{-1}(\bar{\nu}_t^{(1)} - \bar{\nu}_t^{(2)})$, *then* $\eta_t = \xi_t^{(1)}/\xi_t^{(2)}$ *follows the process*

$$
\frac{d\eta_t}{\eta_t} = (\phi_t^{(2)} - \phi_t^{(1)})\, d\tilde{W}_t^{(1)}.
$$

Proof of Lemma 3: Let $\eta_t = g(\xi_t^{(1)}, \xi_t^{(2)})$. Its partial derivatives are $g_{\xi^{(1)}} = 1/\xi^{(1)}$; $g_{\xi^{(2)}} = -\xi_t^{(1)}/\xi_t^{(2)}$; $g_{\xi^{(1)}\xi^{(1)}} = 0$; $g_{\xi_t^{(1)},\xi_t^{(2)}} = -1/(\xi_t^{(2)})^2$; and $g_{\xi_t^{(2)}\xi_t^{(2)}} = 2\xi_t^{(1)}/(\xi_t^{(2)})^3$. Using the dynamics of the real kernels in (9), an application of Ito's lemma implies that

$$
\begin{aligned}
d\eta_t &= \frac{1}{\xi_t^{(2)}} \left(-r_t \xi_t^{(1)} dt - \phi_t^{(1)} \xi_t^{(1)} d\tilde{W}_t^{(1)} \right) \\
&- \frac{\xi_t^{(1)}}{(\xi_t^{(2)})^2} \left(-r_t \xi_t^{(2)} dt - \phi_t^{(2)} \xi_t^{(2)} d\tilde{W}_t^{(2)} \right) + \frac{1}{2} \frac{2\xi_t^{(1)}}{(\xi_t^{(2)})^3} \phi_t^{(2)} {\phi_t^{(2)}}^\top dt - \frac{1}{(\xi_t^{(2)})^2} \xi \phi_t^{(1)} {\phi_t^{(2)}}^\top dt.
\end{aligned}
$$

Therefore,

$$\frac{d\eta_t}{\eta_t} = (-r_t dt - \phi_t^{(1)} d\tilde{W}_t^{(1)}) - (-r_t dt - \phi_t^{(2)} d\tilde{W}_t^{(2)}) + \left(\phi_t^{(2)} \phi_t^{(2)\top} - \phi_t^{(1)} \phi_t^{(2)\top}\right) dt.$$

Now using (7) and the stated condition, and collecting terms completes the proof. ■

Proof of Proposition 1: Suppose agents agree on prices at all dates. By the definition of the market prices of risk, for an asset with current payout flow rate δ_{it} and volatility σ_{it}, the instantaneous risk premium for agent m is

$$\bar{\mu}_{it}^{(m)} + \delta_{it} - r_t = \sigma_{it} \phi_t^{(m)\top}, \tag{A1}$$

which implies that

$$\bar{\mu}_{it}^{(1)} - \bar{\mu}_{it}^{(2)} = \sigma_{it} (\phi_t^{(1)} - \phi_t^{(2)})^\top. \tag{A2}$$

Since (14) and (A2) hold for every asset i at each time t, (15) must hold.

Now suppose that (15) holds. By Lemma 3, $\frac{d\eta_t}{\eta_t} = \sigma_{\eta t}^\top d\tilde{W}_t^{(1)} = \frac{d\varrho_t}{\varrho_t}$. By Corollary 1, η_t is the Radon-Nikodym derivative of $\mathcal{P}_t^{(2)}$ with respect to $\mathcal{P}_t^{(1)}$. Fix an arbitrary time horizon T. Then,

$$E_t^{(2)}[\int_t^T \frac{\xi_s^{(2)}}{\xi_t^{(2)}} \delta_s \, ds] = E_t^{(1)}[\int_t^T \frac{\xi_s^{(2)}}{\xi_t^{(2)}} \frac{\eta_s}{\eta_t} \delta_s \, ds] = E_t^{(1)}[\int_t^T \frac{\xi_s^{(1)}}{\xi_t^{(1)}} \delta_s \, ds], \tag{A3}$$

where the first equality follows from the definition of a Radon-Nikodym derivative and the second from the definition of η_t. Therefore, agents agree on the level of the expected discounted value of fundamentals up to a fixed horizon T. Now, since δ_t and $\xi_t^{(m)}$ are positive, both discounted values are positive and increasing in T. Letting $T \to \infty$, the monotone convergence theorem then implies that the agents agree on the discounted value of fundamentals. Under their respective transversality conditions, they agree on the level of prices, as claimed. ■

Proof of Lemma 2: By individual m's first-order condition for optimal consumption, $c_t^{(m)} = I_m(y_m \xi_t^{(m)})$, where y_m is the Lagrange multiplier for agent m. A straightforward application of Ito's lemma implies that $dc_m = \partial I_m(y_m \xi_t^{(m)})/\partial \xi^{(m)} d\xi^{(m)} + 1/2\, \partial^2 I_m(y_m \xi_t^{(m)})/\partial {\xi^{(m)}}^2 \, (d{\xi^{(m)}}^2)$. Since the optimum condition can also be written as $u_m'(c_t^{(m)}) = y_m \xi_t^{(m)}$, I can write $c_t^{(m)} = I_m(u_m'(c_t^{(m)}))$. Differentiating both sides of the equality implies that $1 = I_m'(y_m \xi_t^{(m)}) u''(c_t^{(m)})$. By the chain rule, $\partial I_m(y_m \xi_t^{(m)})/\partial \xi^{(m)} = I_m'(y_m \xi_t^{(m)}) y_m = y_m / u_m''(c_t^{(m)})$. Using the characterization of individual m's state-price density in (9), I obtain $\sigma_{ct}^{(m)} = -y_m \xi_t^{(m)} / u_m''(c_t^{(m)}) \phi^{(m)} = -u_m'(c^{(m)})/u_m''(c^{(m)})\phi^{(m)}$, which equals the statement of the volatility. Similarly, differentiating consumption twice, I obtain $0 = I_m''(u_m'(c_t^{(m)}) u_m''(c_t^{(m)})^2 + I_m'(u_m'(c_t^{(m)}) u_m'''(c_t^{(m)})$, hence $I_m''(y_m \xi_t^{(m)}) = -u'''(c_t^{(m)})/u_m''(c_t^{(m)})^3$ and $\partial^2 I_m(y_m \xi_t^{(m)})/\partial {\xi^{(m)}}^2 = -u_m'''(c_t^{(m)})/u_m''(c_t^{(m)})^3 y_m^2$. Therefore, $\mu_c^{(m)} = y_m \xi_t^{(m)} / u_m''(c_t^{(m)})(-r_t) + 1/2\, u_m'''(c_t^{(m)})/u_m''(c_t^{(m)})^3 y_m^2 \xi_t^2$, which equals the drift term in the statement. ■

Proof of Proposition 2: To facilitate the analysis of equilibrium, I follow the approach of Cuoco and Hé(1994) to solve for the equilibrium in the effectively incomplete markets model by formulating stochastic weights for the representative agent. Basak (2000) extends the solution method to models with heterogeneous beliefs. Some other important applications of this stochastic weights methodology are in Basak and Cuoco(1998) and Detemple and Serrat(2003). For given weights λ_{1t} and λ_{2t} for the two agents, the representative agent's utility function solves

$$U(c_t; \lambda_{1t}, \lambda_{2t}) = e^{-\rho t} \max_{c_{1t}+c_{2t}=c_t} \lambda_{1t} \frac{c_{1t}^\gamma}{\gamma} + \lambda_{2t} \frac{c_{2t}^\gamma}{\gamma}.$$

Solving this problem gives the equivalent form:

$$U(c_t; \lambda_{1t}, \lambda_{2t}) = e^{-\rho t} \frac{c_t^\gamma}{\gamma} \lambda_{1t} \left(1 + \left(\frac{\lambda_{2t}}{\lambda_{1t}} \right)^{\frac{1}{1-\gamma}} \right)^{1-\gamma}. \quad \text{(A4)}$$

Following the analysis in Basak (2000), I formulate the equilibrium with the weights $\lambda_{1t} = 1/y_1$ and $\lambda_{2t} = \eta_t/y_2$, where y_1 and y_2 are the Lagrange multipliers associated with the budget constraints of the two agents at time 0 in equation (8). It is evident that with these weights, consumption allocations coincide with those of competitive equilibrium, that is, they satisfy $u'(c_t^{(1)})/u'(c_t^{(2)}) = (y_1\xi_t^{(1)})/(y_2\xi_t^{(2)})$, the ratio of individuals' optimality conditions, and by construction the goods market clears.

I define the inverse function of the representative agent's marginal utility of aggregate consumption as $U_c(c_t;\lambda_{1t},\lambda_{2t})^{-1} \equiv I(z_t;\lambda_{1t},\lambda_{2t}) \equiv I_1(\frac{1}{\lambda_{1t}}z_t) + I_2(\frac{1}{\lambda_{2t}}z_t)$. Now, by the special choice of the weights, $\lambda_{1t} = 1/y_1$ and $\lambda_{2t} = \eta_t/y_2$, where $\eta_t = \xi_t^{(1)}/\xi_t^{(2)}$, $I(z_t;y_1,y_2,\eta_t) = I_1(y_1z_t) + I_2(y_2/\eta_t\, z_t)$. Therefore, $I(\xi_t^{(1)};y_1,y_2,\eta_t) = I_1(y_1\xi_t^{(1)}) + I_2(y_2\xi_t^{(2)}) = c_{1t} + c_{2t} = x_t$. Furthermore, since $U_c(\cdot)^{-1} = I(\cdot)$, $I(\xi_t^{(1)}) = x_t$, or $U_c(x_t) = \xi_t^{(1)}$. In addition, $U_c(\cdot)/\eta_t = \xi_t^{(2)}$. Using the form of the representative agent's utility function in (A4) gives the individual consumption processes in (i).

Using the characterization of agent 1's state-price density, $U_c(x) = \xi_t^{(1)}$, for the weights $\lambda_{1t} = 1/y_1$ and $\lambda_{2t} = \eta_t/y_2$, I obtain

$$\xi_t^{(1)} = e^{-\rho t}\, x_t^{\gamma-1}\frac{1}{y_1}\left[1 + \left(\frac{y_1\eta_t}{y_2}\right)^{1/(1-\gamma)}\right]^{1-\gamma}. \tag{A5}$$

Since the drift of $d\xi_t^{(1)}/\xi_t^{(1)}$ equals the negative of the short rate, an application of Ito's lemma along with the equations for the processes x_t and η_t in (2) and (16) implies the riskless rate in (ii).

Market clearing for the consumption good implies that

$$\sigma_{c,qt}^{(1)} + \sigma_{c,qt}^{(2)} = \sigma_{x,1}x_t, \tag{A6}$$

$$\sigma_{c,xt}^{(1)} + \sigma_{c,xt}^{(2)} = \sigma_{x,2}x_t. \tag{A7}$$

Using the volatilities of consumption from Lemma 2 implies that the equilibrium conditions are

$$\frac{1}{a_t^{(1)}}\phi_{qt}^{(1)} + \frac{1}{a_t^{(2)}}\phi_{qt}^{(2)} = \sigma_{x,1}x_t, \tag{A8}$$
$$\frac{1}{a_t^{(1)}}\phi_{xt}^{(1)} + \frac{1}{a_t^{(2)}}\phi_{x}^{(2)} = \sigma_{x,2}x_t. \tag{A9}$$

Equations (15), (A8), and (A9) contain four equations in the four market prices of risk, leading to the unique solution

$$\phi_{qt}^{(1)} = \frac{a_t^{(1)}}{a_t^{(1)}+a_t^{(2)}}\frac{\sigma_{x,2}\,(\bar{\theta}_t^{(1)}-\bar{\theta}_t^{(2)})+\sigma_{q,2}\,(\bar{\kappa}_t^{(2)}-\bar{\kappa}_t^{(1)})}{|\Sigma|} + \frac{a_t^{(1)}a_t^{(2)}}{(a_t^{(1)}+a_t^{(2)})}\sigma_{x,1}x, \tag{A10}$$
$$\phi_{qt}^{(2)} = \frac{a_t^{(2)}}{a_t^{(1)}+a_t^{(2)}}\frac{\sigma_{x,2}\,(\bar{\theta}_t^{(2)}-\bar{\theta}_t^{(1)})+\sigma_{q,2}\,(\bar{\kappa}_t^{(1)}-\bar{\kappa}_t^{(2)})}{|\Sigma|} + \frac{a_t^{(1)}a_t^{(2)}}{(a_t^{(1)}+a_t^{(2)})}\sigma_{x,1}x, \tag{A11}$$
$$\phi_{xt}^{(1)} = \frac{a_t^{(1)}}{a_t^{(1)}+a_t^{(2)}}\frac{\sigma_{x,1}\,(\bar{\theta}_t^{(2)}-\bar{\theta}_t^{(1)})+\sigma_{q,1}\,(\bar{\kappa}_t^{(1)}-\bar{\kappa}_t^{(2)})}{|\Sigma|} + \frac{a_t^{(1)}a_t^{(2)}}{(a_t^{(1)}+a_t^{(2)})}\sigma_{x,2}x, \tag{A12}$$
$$\phi_{xt}^{(2)} = \frac{a_t^{(2)}}{a_t^{(1)}+a_t^{(2)}}\frac{\sigma_{x,1}\,(\bar{\theta}_t^{(1)}-\bar{\theta}_t^{(2)})+\sigma_{q,1}\,(\bar{\kappa}_t^{(2)}-\bar{\kappa}_t^{(1)})}{|\Sigma|} + \frac{a_t^{(1)}a_t^{(2)}}{(a_t^{(1)}+a_t^{(2)})}\sigma_{x,2}x. \tag{A13}$$

Notice that the market prices of risk depend on beliefs of investors of each type through the conditional means of each of the state variables, as well as on their risk aversions and consumption levels through the coefficients a_m. For the case of CRRA preferences, $a_t^{(m)} = -u''[c^{(m)}]/u'[c^{(m)}] = (1-\gamma)/c_t^{(m)}$. Substituting these into equations (A10) through (A13) implies (iii). ■

Proof of Corollary 2: Equations (20) and (22) through (25), imply that $\phi_t^{(1)}\,\frac{c_t^{(1)}}{x_t} + \phi_t^{(2)}\,\frac{c_t^{(2)}}{x_t} = (1-\gamma)\sigma_x$. Multiplying (13) by $\frac{c^{(m)}}{x}$ for each *m*, adding across agents, and using the above equality completes the proof. ■

Proof of Proposition 3: Proposition 2 provides consumption processes and SPDs for each type as functions of k that are consistent with utility maximization, market clearing, and the structure of the state variables in the economy. Proposition 1 (agreement of values) implies that for all k,

$$\begin{aligned}\sum_{m=1}^{2} X^{(m)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k) &= E^{(1)}[\int_0^\infty \frac{\xi_s^{(1)}}{\xi_t^{(1)}}\,c_s^{(1)}\,ds] + E^{(2)}[\int_0^\infty \frac{\xi_s^{(2)}}{\xi_t^{(2)}}\,c_s^{(2)}\,ds] \\ &= E^{(1)}[\int_0^\infty \frac{\xi_s^{(1)}}{\xi_t^{(1)}}\,c_s^{(1)}\,ds] + E^{(1)}[\int_0^\infty \frac{\xi_s^{(1)}}{\xi_t^{(1)}}\,c_s^{(2)}\,ds] \\ &= E_t^{(1)}[\int_t^\infty \frac{\xi_s^{(1)}}{\xi_t^{(1)}}\,x_s\,ds] = E_t^{(1)}[\int_t^\infty \frac{\xi_s^{(1)}}{\xi_t^{(1)}}\,(e^{(1)}+e^{(2)})\,x_s\,ds] \\ &= E_t^{(1)}[\int_t^\infty \frac{\xi_s^{(1)}}{\xi_t^{(1)}}\,e^{(1)}\,x_s\,ds] \,+\, E_t^{(2)}[\int_t^\infty \frac{\xi_s^{(2)}}{\xi_t^{(2)}}\,e^{(2)}\,x_s\,ds] \\ &= \sum_{m=1}^{2} V^{(m)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k),\end{aligned}$$

where the second equality holds because agents agree on the value of agent 2's consumption flow, the third and fourth from consumption flows in (20) and Assumption 5, respectively, and the fifth because the agents agree on the value of agent 2's endowment stream. Given this equality, I have that when $X^{(1)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k) = V^{(1)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k)$, $X^{(2)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k) = V^{(2)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k)$ as well. Next notice that $X^{(1)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k) - V^{(1)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k)$ is a continuous function of k, which is positive at $k=0$ and negative as $k\to\infty$ since by (20), $c_s^{(1)}(k=0) - e^{(1)}\,x_s > 0, \forall s$ and $c_s^{(1)}(k\to\infty) - e^{(1)}\,x_s < 0, \forall s$. Therefore, there exists a $k^*>0$ such that $X^{(1)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k^*) = V^{(1)}(\pi^{*(1)},\pi^{*(2)},\eta_0;k^*)$, and hence equilibrium exists.

The derivative $\frac{\partial(X_0^{(1)}-V_0^{(1)})}{\partial k}$ equals $E^{(1)}[\int_0^\infty A_s\cdot B_s\cdot(C_s-D_s)\;ds]$, in which $A_s = (k\,\eta_s^{\frac{1}{1-\gamma}}+1)^{-\gamma-1}$, $B_s = (k\,\eta_0^{\frac{1}{1-\gamma}}+1)^{\gamma-2}$, $C_s = \eta_s^{\frac{1}{1-\gamma}}(e^{(m)}\,(\gamma-1)[k\,\eta_s^{\frac{1}{1-\gamma}}+1]-\gamma)$, and $D_s = ([e^{(m)}\cdot(\gamma-1)+1]\,k\,\eta_s^{\frac{1}{1-\gamma}} + (e^{(m)}-1)(\gamma-1))\eta_0^{\frac{1}{1-\gamma}}$. It is straightforward to see that $A_s > 0$, $B_s > 0$, $C_s < 0$, and $D_s > 0$ for all s when $0 \le \gamma < 1$, and hence the derivative is strictly negative, which implies that k^* is unique. ■

Appendix B: Stocks in Positive Net Supply

In the model discussed so far, stocks have been in zero net supply. As footnote 4 claims, with the alternative assumption of stocks being in "small" positive supply the equilibrium premium and riskless rate are very similar. In this context total consumption in any period equals $x_t + q_t$, the sum of dividends and output in the economy produced from resources not financed by public equity. The additional complication is the introduction of another state variable, $q_t/(x_t + q_t)$, the dividend share of total output. I am once again able to solve for the riskless rate and market prices of risk in this alternative economy. Similar to (21), I find that the riskless rate is

$$\begin{aligned} r_t &= \rho - \frac{1}{2}\,(2-\gamma)\,(1-\gamma)\left(\frac{x_t}{x_t+q_t}\sigma_x + \frac{q_t}{x_t+q_t}\sigma_q\right)^2 \\ &+ \frac{q_t}{x_t+q_t}\,\frac{1-\gamma}{1+k\,\eta_t^{\frac{1}{1-\gamma}}}\left(\bar{\theta}_t^{(1)} + k\,\eta_t^{\frac{1}{1-\gamma}}\,\bar{\theta}_t^{(2)}\right) + \frac{x_t}{x_t+q_t}\,\frac{1-\gamma}{1+k\,\eta_t^{\frac{1}{1-\gamma}}}\left(\bar{\kappa}_t^{(1)} + k\,\eta_t^{\frac{1}{1-\gamma}}\,\bar{\kappa}_t^{(2)}\right) \\ &- \frac{\gamma\,k\,\eta_t^{\frac{1}{1-\gamma}}\,[(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})\sigma_x - (\bar{\kappa}_t^{(1)} - \bar{\kappa}_t^{(2)})\sigma_q]^2}{2\,(1-\gamma)\left(1+k\,\eta_t^{\frac{1}{1-\gamma}}\right)^2 |\Sigma|^2}. \end{aligned} \tag{B1}$$

As before, the terms have similar interpretation, where the term for precautionary savings must now incorporate the weighted average of volatility of the two processes, and the term for the wealth effect must incorporate the expected growth of dividends. As $q_t/(x_t + q_t) \to 0$, I once again obtain (21). Notably, the speculative risk component (last term) is identical in the two expressions.

Similarly, solving for the market prices of risk (I only provide the equation for $\phi_q^{(1)}$),

$$\phi_q^{(1)} = (1-\gamma)\left(\frac{q_t}{q_t+x_t}\sigma_{q,1} + \frac{x_t}{q_t+x_t}\sigma_{x,1}\right) + \frac{k\,\eta_t^{\frac{1}{1-\gamma}}}{1+k\,\eta_t^{\frac{1}{1-\gamma}}}\,\frac{(\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)})\sigma_{x,2} + (\bar{\kappa}_t^{(2)} - \bar{\kappa}_t^{(1)})\sigma_{q,2}}{|\Sigma|}, \tag{B2}$$

which is identical to (22) with the sole exception that the market price of aggregate risk contains the weighted average of dividend and consumption volatilities. Once again, the price of the speculative risk component is the same.

Solving for asset prices would be similar, but I would have to incorporate the additional state variable, which would be tedious but still possible using projection methods. Nonetheless, since the equity premium puzzle can be restated as the difficulty of attaining a low riskless return and high Sharpe ratio, it is straightforward to assess whether the change in assumption significantly affects my results. Calibrating my economy to aggregate dividends and output I find that the ratio $q_t/(q_t + x_t)$ is on average about 2.6%, and for the case $\gamma = 0.5$, leads to an average riskless rate that is higher only by four basis points. The values of the four market prices of risk also differ by similar amounts relative to the zero net supply case. Therefore, the alternative assumption does not affect Sharpe ratios under the measures of the different agents by large amounts.

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